Differential and Integral Calculus 1 - MS-A0111
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Exam
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Remember to write your name and student number on the solutions you return. And note that this sheet has a second page! You are not allowed to use a calculator, tables or notes.

Every problem carries an equal weight. Similarly, every part of a problem carries an equal weight.

Explain the reasoning behind your solutions, do not just write the final result.

PROBLEM 1 Compute the following limits:

$$(a) \quad \lim_{x\to 0} \frac{(\cos x)^2 + x^2 - 1}{x^4} \qquad (b) \quad \lim_{x\to 0} \frac{e^{3x} - \sin(3x) - 1}{\ln(1-2x)}.$$

PROBLEM 2 Consider the function $f(x) = (\sin(\cos x))^2 + (\cos(\cos(x)))^2$.

a) Compute the derivative of f using only famous derivatives and differentiation rules, without using trigonometric formulas.

b) What does this tell you about the function f? What is its value?

PROBLEM 3 Compute the integrals

$$\text{(a)} \quad \int x^2 \sin x \, dx \qquad \text{(b)} \quad \int_0^{\sqrt{8}} \frac{x^3}{\sqrt{x^2+1}} \, dx.$$

PROBLEM 4 Find all the solutions to $y' + 2y = 3$.

PROBLEM 5 Consider the function $f(x) = \frac{\sin(2x)}{16}$.

a) Compute the third Taylor polynomial P_3 for f about $a = \pi/2$.

b) If you approximate $f(\frac{\pi}{2} + 1)$ with $P_3(\frac{\pi}{2} + 1)$, is the error smaller than $\frac{1}{20}$? Explain why.

PROBLEM 6 Consider the function $f(x) = \frac{e^x - 2}{1 - e^x}$.

a) For what values of x is the function defined? Compute the limits

$$\lim_{x \to -\infty} f(x), \qquad \lim_{x \to +\infty} f(x),$$

and at the points a where f is not defined, compute

$$\lim_{x \to a^-} f(x), \qquad \lim_{x \to a^+} f(x).$$

b) Compute the first derivative of f and study its sign: where is it positive, negative, zero? Where does f increase/decrease?

c) Compute the second derivative of f and study its sign: where is it positive, negative, zero? Where is f convex (happy)/concave (sad)?

d) Use the information above to sketch the graph of f.

Formulas.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + O(x^{n+1}) \qquad \text{as } x \to 0$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2}) \qquad \text{as } x \to 0$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3}) \qquad \text{as } x \to 0$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + O(x^{n+1}) \qquad \text{as } x \to 0$$

$$\sin(\pi) = 0 = 1 + \cos(\pi)$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$