ELEC-E8116 Model-based control systems
Full exam. 22. 12. 2021 / Solutions

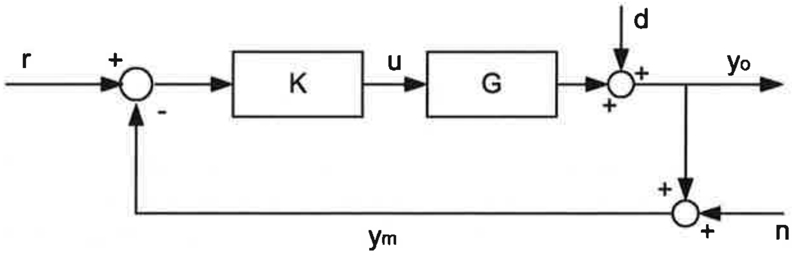
- *Write the name of the course, your name and student number to each answer sheet.*
- *There are five (5) problems and each one must be answered.*
- *Read the instructions in a separate file (Instructions), which is available in the Exam Assignment and which has also been published in advance.*
- *In problem 0 sign with your name (typesetting is enough if you use computer document) in which you assure that you follow the exam regulations.*

Each problem 1-5 gives the maximum of 6 points.

Note 1: Solutions obtained by computer are not presented and not accepted. Computer can be used in verification, but that is only for yourself to check the results, if needed.
Note 2: In many questions below you have to explain something (e.g. concepts). It is then not enough that you take some formulas, expressions or text from the course open book material; instead, you have to answer the questions in your own words such that the reviewer can see that you have really understood the issue. The answers do not have to be long, but they must be clear to the reader.

0. Write your signature confirming that you follow the exam rules.

1. Consider the multivariable closed-loop control configuration



Write the equations describing the system and identify

a. closed-loop transfer function (1p)
b. sensitivity function (1p)
c. complementary sensitivity function (1p)

Write expressions for the output variable y_0, control variable u and error variable $e = r - y_0$. What conditions to the above functions (a-c) should be set, in order the system to operate "well"? (3p)

2. For the system

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = x_0$$

calculate the control law and optimal cost, when the criterion to be minimized is

$$J = \frac{1}{2}x(1)^2 + \int_0^1 u^2(\tau)d\tau$$ (3p+3p)

3. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

a. Calculate the *singular values* of the matrix. (2p)
b. Explain, what *singular value decomposition* means. How can it be used to analyse multivariable systems? (4p)

4. Explain briefly the following concepts (1p each):

a. Principle of Optimality
b. Small Gain Theorem
c. Conservative control law
d. Smith predictor
e. Loop transfer recovery
f. Robust stability

5. Consider a multivariable system with the transfer function matrix

$$G(s) = \begin{bmatrix} \frac{a}{s+1} & \frac{b}{s+2} \\ \frac{c}{s+1} & \frac{d}{s+1} \end{bmatrix},$$ where the parameters a, b, c, d are real constants, not all zero.

a. Calculate the poles and zeros of the system. Is it possible to choose the values of parameters such that the system would be unstable? (2p)
b. Determine conditions of the parameters such that the minimal realization of the system will have two states. Show an example of a suitable parameter set. (2p)
c. Determine conditions for the parameters such that the system will have an RHP zero. Show an example of a suitable parameter set. (2p)

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1. $y_0 = d + GK[r - (y_0 + n)] = d + GKr - GKy_0 - GKn$

Chap 3 in lecture slides

$\Rightarrow (I + GK)y_0 = d + GKr - GKn$

$\Rightarrow y_0 = (I+GK)^{-1}d + (I+GK)^{-1}GKr - (I+GK)^{-1}GKn$

$= (I+L)^{-1}d + (I+L)^{-1}Lr - (I+L)^{-1}Ln$

$= Sd + Tr - Tn$

where $L = GK$ loop transfer function

$S = (I+L)^{-1}$ sensitivity function

$T = (I+L)^{-1}L$ complementary sensitivity function

Note that this is a 1 DOF system, so that $G_{cl} = T$

$e = r - y_0 = r - Sd - Tr + Tn = (I-T)r - Sd + Tn$

$= Sr - Sd + Tn$

$(S + T = I)$

$u = K[r - (y_0 + n)] = Kr - Ky_0 - Kn$

$= Kr - KSd - KTr + KTn - Kn$

$= K(I-T)r - KSd + K(T-I)n$

$= KSr - KSd - KSn$

Conditions for good performance:

S small (r, d)

T large $(\approx I)$: r

T small (n)

2. $-\dot{S}(t) = A^T S + SA - SBR^{-1}B^T S + Q \quad , t \leq t_f$

Boundary condition $S(t_f)$

Chap 5 in lecture slides

$K = R^{-1}B^T S$

$u^* = -Kx$

$J^*(t_0) = \frac{1}{2} x^T(t_0) S(t_0) x(t_0)$

Problem. $\dot{x}(t) = x(t) + u(t), \; x(0) = x_0$

$$J = \frac{1}{2} x(1)^2 + \int_0^1 u^2(\tau) d\tau = \frac{1}{2} x(1)^2 + \frac{1}{2} \int_0^1 2u^2(\tau) d\tau$$

- Scalar system, $t_0 = 0$, optimization horizon $0 \to 1 \; (= t_f)$

Final state free (with cost)

$A = 1, \; B = 1, \; S(t_f) = S(1) = 1, \; Q = 0, \; R = 2$

Riccati:

$$-\dot{s}(t) = s(t) + s(t) - s(t) \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot s(t) + 0$$

$\Rightarrow -\dot{s}(t) = 2s(t) - \frac{1}{2} s(t)^2 \quad , s(1) = 1$

$\Rightarrow \frac{ds}{dt} = \frac{1}{2}s^2 - 2s = s\left(\frac{1}{2}s - 2\right)$

$\Rightarrow \frac{ds}{s(\frac{1}{2}s - 2)} = dt \Rightarrow \frac{2\,ds}{s(s-4)} = dt \Rightarrow \left(\frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-4}\right) ds = dt$

$\Rightarrow \int \left(\frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-4}\right) ds = \int dt \Rightarrow -\frac{1}{2} \ln|s| + \frac{1}{2} \ln|s-4| = t + C_1$

$\Rightarrow \frac{1}{2} \ln\left|\frac{s-4}{s}\right| = t + C_1 \Rightarrow \ln\left|\frac{s-4}{s}\right| = 2t + 2C_1$

$$\Rightarrow \frac{s-4}{s} = \pm e^{2t+2C_1} = \pm e^{2t} e^{2C_1} = C_2 e^{2t}$$

$$\Rightarrow s-4 = C_2 e^{2t} s \Rightarrow s\left(1 - C_2 e^{2t}\right) = 4 \Rightarrow$$

$$\Rightarrow s(t) = \frac{4}{1-C_2 e^{2t}}$$

Boundary condition $s(1) = 1 \Rightarrow s(1) = \frac{4}{1-C_2e^2} = 1$

$$\Rightarrow 1 - C_2 e^2 = 4 \Rightarrow C_2 = \frac{-3}{e^2} = -3e^{-2}$$

$$\Rightarrow s(t) = \frac{4}{1+3e^{-2}e^{2t}} = \frac{4}{1+3e^{2t-2}} = \frac{4}{1+3e^{2(t-1)}}$$

$$u^*(t) = -R^{-1}B^T s x = -\frac{1}{2} \cdot 1 \cdot \frac{4}{1+3e^{2(t-1)}} x(t) = -\frac{2}{1+3e^{2(t-1)}} x(t)$$

$$J^*(0) = \frac{1}{2} x_0^2 s(0) = \frac{1}{2} \cdot \frac{4}{1+3e^{-2}} x_0^2 = \frac{2}{1+3e^{-2}} x_0^2$$

3.

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ The singular values are

$$\sigma = \sqrt{\lambda(A^* A)} \approx \begin{cases} 0.3660 \\ 5.4650 \end{cases}$$

$$\left[A^* A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}, \quad \lambda(A^* A) = \begin{cases} 0.1339 \\ 29.8661 \end{cases}, \quad \sqrt{\lambda(A^* A)} = \begin{matrix} 0.3660 \\ 5.4650 \end{matrix} \right]$$

b) All square matrices (real or complex) have an SVD

$$A = U \Sigma V^*$$

where U and V are unimodular ($U^* = U^{-1}$, $V^* = V^{-1}$) and Σ is a real matrix with singular values of A in the main diagonal.

Further $AV = U \Sigma V^* V = U \Sigma$

$$\Rightarrow \quad A[v_1 \vdots v_2 \vdots \cdots \vdots v_n] = [\sigma_1 u_1 \vdots \sigma_2 u_2 \vdots \cdots \vdots \sigma_n u_n]$$

where v_i are the columns of V and u_i are the columns of U.

So $\boxed{A v_i = \sigma_i u_i, \quad i = 1, 2, \ldots, n}$ $(*)$

The singular value σ_i gives the "strength" of input i in the input direction v_i to output i in the output direction u_i of system A.

That can be used in analysing interconnections in multivariable systems. Also, SVD can be used in the design of decoupling controllers.

4. a) See lecture slides, Chapter 5.
b) — " —, Chapter 1.
c) A control law that fulfils one or more design criteria, but is still ineffective generally. For example, it can be too slow (lazy) to be practically usable.
A control law can be robustly stable, but if it does not meet any criterion of robust performance, it can be really bad. An ultimate example: controller that produces only a zero signal to the process. The closed loop might be stable, but the performance is definitely not what will be expected.

A more practical example: The Mixed Sensitivity Design can lead to slow responses, if the weight functions have been chosen too modestly.

d) See Exercises 10, Problem 2.
e) See lecture slides, Chapter 8.
f) — " —, Chapter 3.

5.

$$G(s) = \begin{bmatrix} \frac{a}{s+1} & \frac{b}{s+2} \\ \frac{c}{s+1} & \frac{d}{s+1} \end{bmatrix}$$

a) Minors: $\frac{a}{s+1}$, $\frac{b}{s+2}$, $\frac{c}{s+1}$, $\frac{d}{s+1}$, $\frac{ad}{(s+1)^2} - \frac{bc}{(s+1)(s+2)}$

$$= \frac{ad(s+2) - bc(s+1)}{(s+1)^2(s+2)} = \frac{(ad-bc)s + 2ad - bc}{(s+1)^2(s+2)}$$

Pole polynomial: $p(s) = (s+1)^2(s+2)$, 3 poles, $-1, -1, -2$

Asymptotically stable irrespective of parameter values.

Zero polynomial: $z(s) = (ad-bc)s + 2ad - bc$

One zero $\frac{bc-2ad}{ad-bc}$ provided that $ad-bc \neq 0$.

b) Generally, the minimal realization would have 3 states.
But if there is a pole-zero-cancellation in the largest minor

either $\overset{1}{\begin{cases} ad-bc = 1 \\ 2ad-bc = 1 \end{cases}}$ or $\overset{2}{\begin{cases} ad-bc = 1 \\ 2ad-bc = 2 \end{cases}}$

then the minimal realization has only 2 states.

1: $-ad = 0$, $bc = -1$ 2: $-ad = -1$, $bc = 0$

$a = 0 \vee d = 0$, $b = -\frac{1}{c}$, $c \neq 0$ $b = 0 \vee c = 0$, $a = \frac{1}{d}$, $d \neq 0$

In fact, it is enough that $a = 0 \vee d = 0 \vee b = 0 \vee c = 0$

Example: $a = 0$, $b = 2$, $c = 3$, $d = 4$.

c. $\frac{bc - 2ad}{ad - bc} > 0$, $ad - bc \neq 0$

$$\frac{bc-2ad}{ad-bc} = \frac{bc-ad-ad}{-(bc-ad)} = -\left[1 - \frac{ad}{bc-ad}\right]$$

$$= \frac{ad}{bc-ad} - 1 > 0$$

For example: $a=1$, $b=2$, $c=3$, $d=4$

$$\frac{bc-2ad}{ad-bc} = \frac{6-8}{4-6} = \frac{-2}{-2} = 1$$ RHP zero 1.