ELEC-E8116 Model-based control systems
Intermediate exam 1. 27. 10. 2021 / Solutions

- *Write the name of the course, your name and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *Read the instructions in a separate file (Instructions), which is available in the Exam Assignment and which has also been published in advance.*
- *In problem 0 sign with your name (typesetting is enough if you use computer document) in which you assure that you follow the exam regulations.*

Each problem 1-3 gives the maximum of 5 points.

Note: Solutions obtained by computer are not presented and not accepted. Computer can be used in verification, but that is only for yourself to check the results, if needed.

0. Write your signature.

1. Explain briefly the following concepts (max 1p. each)

- Singular value decomposition and singular values
- Input and output directions of a multivariable system
- Minimal realization of a multivariable transfer function
- Internal stability
- Robust stability

Solution:

For any real or complex *mxn* matrix the exists the *singular value decomposition*

$$G = U\Sigma V^*$$

where the *mxm* and *nxn* matrices *U* and *V* are *unitary* ($UU^* = U^*U = I_m, VV^* = V^*V = I_n$) and the real *mxn* matrix Σ has the singular values in its main diagonal. If *G* is complex, then *U* and *V* are also complex, otherwise real. The singular values can also be calculated from

$$\sigma = \sqrt{\lambda(G^*G)} = \sqrt{\lambda(GG^*)}$$

where the non-zero eigenvalues are the same in both expressions.

By using the SVD it is easy to show that

$$Gv_i = \sigma_i u_i$$

where v_i are the columns of matrix *V*, u_i are the columns of matrix *U* and σ_i is the corresponding singular value. The singular value thus gives the strength of *input direction* v_i in the *output direction* u_i. It therefore describes the behaviour of the system in terms of input channels, output channels and system gain (norm).

Minimal realization of a system transfer function is a state-space realization (representation) of it by using the minimum number of state variables. Then the representation is both controllable and reachable. For SISO systems the minimal realization can easily be determined by first cancelling the same polynomials from numerator and denominator (pole-zero cancellations). Then the resulting input-output system is both controllable and observable.

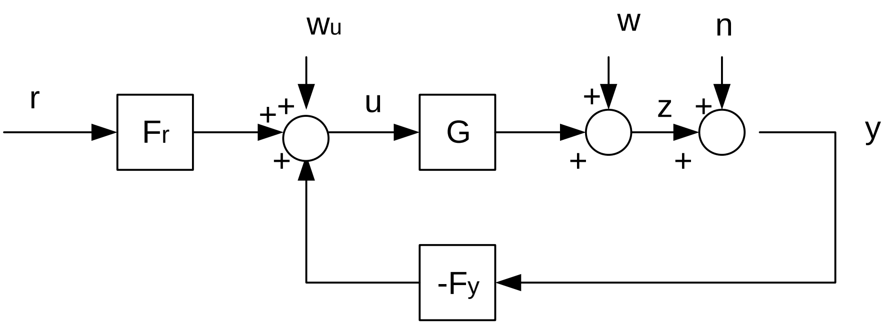
For MIMO systems the determination of minimum number of states is not trivial, since it is not easy to determine possible pole-zero cancellations. Help is given by *Karcanias theorems* (giving pole and zero polynomials that correspond the minimal realization).

The system is internally stable when no finite input signal (reference, disturbance) can lead to some output signal to grow without limit.

The system is robustly stable, when it remains stable for all values of the possible uncertainty in the system.

Note for problems 2 and 3: When writing equations with matrices remember that a square matrix can have inverse A^{-1} (if exists), but you cannot divide by a matrix ($1/A$ is illegal operation for a matrix). Also note that for matrices $AB \neq BA$ except for some rare special cases.

2. Consider a **multivariable** control configuration in the below figure, where signal y is m-dimensional and signal u n-dimensional (m and n are positive integers).



a. What are the dimensions of signals r, w_u, w, n and matrices G, F_y, F_r? (1 p.)

b. Give the condition by which the 2 DOF (two degrees-of-freedom) control configuration in the figure changes into a 1 DOF configuration. Draw a figure. (2 p.)

c. From the 1 DOF configuration identify the *loop transfer function*, *closed loop transfer function*, *sensitivity function* and *complementary sensitivity function*. Then answer: If the sensitivity function is known, can you calculate the loop transfer function? If the answer is yes, show the resulting formula for L. (2 p.)

Solution:

a.
$$\dim w = \dim n = \dim z = \dim y = \dim r = m$$
$$\dim u = \dim w_u = n$$
$$\dim G = mxn$$
$$\dim F_y = \dim F_r = nxm$$

b. *Fy* =*Fr* (=*K*). Figure: just remove the blocks *Fy* , *Fr* and set *K* in front of *G,* inside the loop.

c.
$$L = GK$$
$$G_c = (I+L)^{-1}L$$
$$S = (I+L)^{-1}$$
$$T = (I+L)^{-1}L = G_c \quad \text{(1 DOF)}$$

$$S = (I+L)^{-1} \Rightarrow I + L = S^{-1} \Rightarrow L = S^{-1} - I$$

3. **a**. Let G and F_y be matrices of dimensions m x n and n x m respectively (m and n are positive integers). Calculate and try to get as simple result as possible to

$$(I + GF_y)^{-1}GF_y - GF_y(I + GF_y)^{-1} = ?$$

where the inverse matrices are assumed to exist and I:s are identity matrices of appropriate dimensions. You may use a well-known matrix identity without proving it. (2 p.)

b. Explain in your own words the concept *bandwidth* from control viewpoint. Then explain what *loop shaping* in control means. (Relate the two concepts to each other in your answer). (3 p.)

Solution:

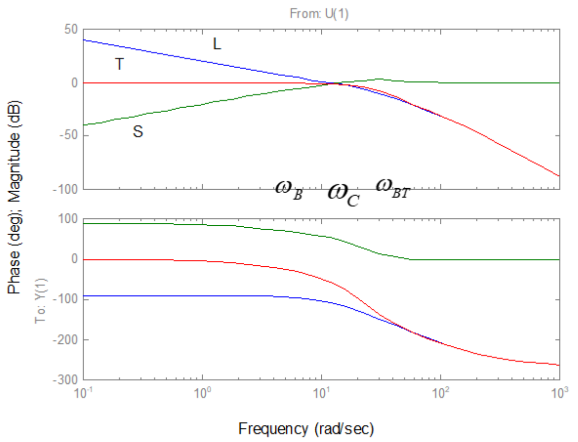
a. The two expressions are equal, and therefore the result is the zero matrix $0_m = 0_{mxm}$. To see that use the push-through rule

$$(I + GF_y)^{-1}GF_y = G(I + F_yG)^{-1}F_y = GF_y(I + GF_y)^{-1}$$

A direct way would be to take $A = GF_y$, $B = I_m$ in

$$(I + AB)^{-1}AB = A(I + BA)^{-1}$$

b. From the lecture slides



Gain crossover ω_C

S -3 dB from ω_B below

T -3 dB from ω_{BT} above

Bandwidth in control means the frequency range for which control is effective. It means the range in which the gain of the frequency response has not started to decrease too much (the system can react to sinusoidal inputs). Bandwidth is usually given by ω_c, ω_B or ω_{BT} as shown in the figure. From exercises: when phase margin (PM) is less than 90 degrees, then it holds $\omega_B < \omega_c < \omega_{BT}$ as in the figure. In control engineering the value ω_B is commonly used to denote bandwidth, whereas in other disciplines ω_{BT} is more common.

Loop shaping means that you set the control target by giving specifications for the frequency response, e.g. for L, S, T , bandwidth (or some of them) and then try to synthesize a controller to meet the desired specification.