Proof of Exercise 1 Demo

Show that the eigenvalues of a real valued symmetric matrix are always real valued.

Let A be a symmetric real valued $p \times p$ matrix $\left(A = A^\top\right)$. Note that, if the symmetry condition is dropped, A can have complex valued eigenvalues and -vectors. Let λ_i be the ith eigenvalue and v_i the corresponding eigenvector of A.

Definition 1 *A scalar λ_i is called an eigenvalue of the $p \times p$ matrix A if there is a nontrivial solution v_i to*

$$Av_i = \lambda_i v_i,$$

where v_i is called an eigenvector corresponding to the eigenvalue λ_i.

Here, trivial solutions are obtained if $v_i = 0$ (zero vector) since every scalar λ_i would then satisfy the equation above. First, we take the complex conjugate from both sides

$$\begin{aligned} \overline{(Av_i)} &= \overline{(\lambda_i v_i)} \\ \Rightarrow A\bar{v}_i &= \bar{\lambda}_i \bar{v}_i, \end{aligned}$$

since A is real valued. Then, we multiply the above with $v_i^\top$ from the left side

$$\begin{aligned} v_i^\top A\bar{v}_i &= v_i^\top \bar{\lambda}_i \bar{v}_i \\ v_i^\top A^\top \bar{v}_i &= v_i^\top \bar{\lambda}_i \bar{v}_i \\ (Av_i)^\top \bar{v}_i &= \bar{\lambda}_i v_i^\top \bar{v}_i \\ \lambda_i v_i^\top \bar{v}_i &= \bar{\lambda}_i v_i^\top \bar{v}_i \\ \Rightarrow \left(\lambda_i - \bar{\lambda}_i\right) v_i^\top \bar{v}_i &= 0. \end{aligned}$$

Note that $v_i^T I \bar{v}_i = \langle v_i, v_i \rangle$ is the canonical Hermitian inner product which is ≥ 0 and $\langle v_i, v_i \rangle = 0$ if and only if $v_i = 0$. By definition, the eigenvectors cannot be zero vectors. Hereby, $\lambda_i = \bar{\lambda}_i$ which implies that λ_i has to be real valued.