Proof of Exercise 2.2

Let A be a symmetric matrix with distinct eigenvalues. Show that the eigenvector matrix of A is orthogonal.

Let λ_i be the ith eigenvalue and v_i the corresponding eigenvector of a $p \times p$ matrix A. The goal is to show that $v_i^\top v_j = 0$, $i \neq j$. We can order the eigenvalues such that $\lambda_1 > \lambda_2 > \ldots > \lambda_p$. The eigenvalues and -vectors satisfy:

$$\begin{cases} Av_i = \lambda_i v_i \\ Av_j = \lambda_j v_j. \end{cases}$$

First, we multiply the first equation with $v_j^\top$ from the left side,

$$\begin{aligned} & v_j^\top A v_i = \lambda_i v_j^\top v_i \\ & v_j^\top A^\top v_i = \lambda_i v_j^\top v_i \\ & \left(A v_j\right)^\top v_i = \lambda_i v_j^\top v_i \\ & \lambda_j v_j^\top v_i = \lambda_i v_j^\top v_i \\ & \Rightarrow \left(\lambda_j - \lambda_i\right) v_j^\top v_i = 0. \end{aligned}$$

Since $\lambda_i \neq \lambda_j$, vectors v_j and v_i have to be orthogonal: $v_j^T v_i = 0$, $i \neq j$. Hereby, the eigenvector matrix V of A satisfies $VV^T = I$, if we choose the eigenvectors of A that have length 1.