Proof of Exercise 6 Demo

Let Z be the matrix defined as in the lecture slides. Then denote the PCA performed on the row profiles as V and on the column profiles as W (scaled and shifted). The matrices are defined the following way:

$$V = Z^\top Z$$
$$W = ZZ^\top.$$

Show that V and W have the same nonzero eigenvalues. Furthermore, show that the following relation holds for the normed eigenvectors that correspond to nonzero eigenvalues:

$$v_i = \frac{1}{\sqrt{\lambda_i}} Z^\top w_i$$
$$w_i = \frac{1}{\sqrt{\lambda_i}} Z v_i,$$

where v_i is the i:th normed eigenvector of V and w_i is the i:th normed eigenvector of W.

First, we show that $Z^\top Z$ and $ZZ^\top$ have the same eigenvalues. From the definition of an eigenvector and -value:

$$\begin{cases} V v_i = Z^\top Z v_i = \lambda_i v_i \\ W w_i = ZZ^\top w_i = \mu_i w_i \end{cases}$$

Multiply the first equation with Z from the left side and note that $V = Z^\top Z$,

$$\Rightarrow ZV v_i = ZZ^\top Z v_i = \lambda_i Z v_i.$$

Thus

$$ZZ^\top (Z v_i) = \lambda_i (Z v_i)$$
$$\Rightarrow ZZ^\top v_i^* = \lambda_i v_i^*, \qquad \text{where } v_i^* = Z v_i.$$

Hereby, λ_i is the eigenvalue of $ZZ^\top = W$ with the eigenvector $Z v_i$. The squared length of the eigenvector is given by

$$||Z v_i||_2^2 = (Z v_i)^\top (Z v_i) = v_i^\top Z^\top Z v_i = \lambda_i v_i^\top v_i = \lambda_i.$$

Hence

$$w_i = \frac{1}{\sqrt{\lambda_i}} Z v_i.$$

The same proof goes to the other direction also.