EPÄOLEELLISET INTEGRAALIT

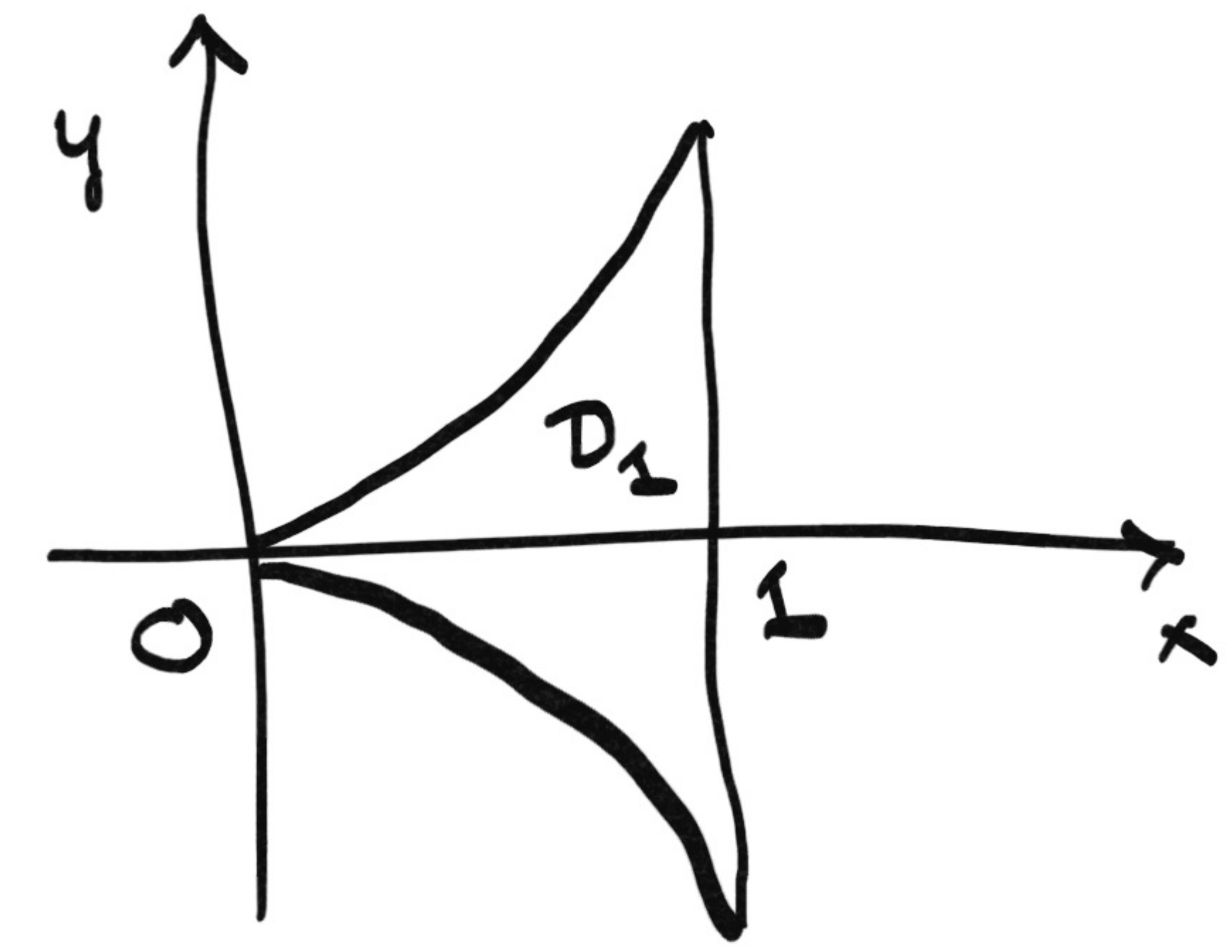
Erikoistapaukset:

(A) rajoitettu funktio (positiivinen) rajoittamaton alue

(B) Rajoittamaton funktio rajoitetussa alueessa

$$f(x,y) = 1/x^2$$

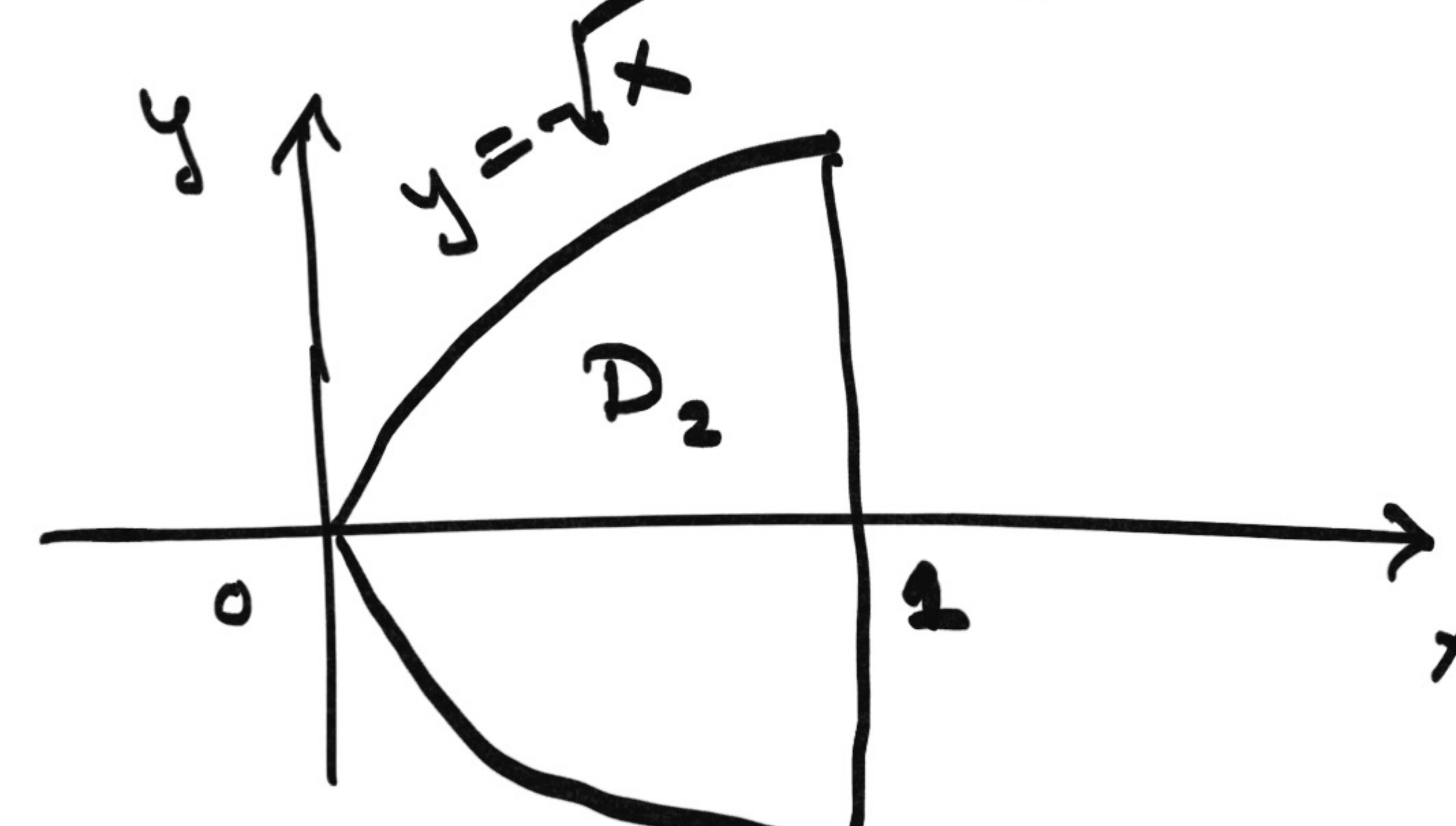
$$D_1 = \{(x,y) \mid 0 \le x \le 1, \; |y| \le x^2\}$$



$$\iint_{D_1} \frac{1}{x^2}\, dA = \int_0^1 \int_{-x^2}^{x^2} \frac{1}{x^2}\, dy\, dx$$

$$= \int_0^1 2\, dx = 2$$

$$D_2 = \{(x,y) \mid 0 \le x \le 1, \; |y| \le \sqrt{x}\}$$



$$\iint_{D_2} \frac{1}{x^2}\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{x^2}\, dy\, dx$$

$$= \int_0^1 2x^{-3/2}\, dx = \infty$$

MUUTTUJAN VAIHTO

Sijoitusmenettely: (Ketjusääntö)

$$I = \int_a^b f(x)\, dx$$

$$x = g(u) \quad ; \quad dx = g'(u)\, du$$

Määrätyssä integraalissa

$$a = g(\alpha) \quad , \quad b = g(\beta)$$

$$I = \int_\alpha^\beta f(g(u))\, g'(u)\, du$$

Riemann:

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x_i$$

<u>Ympyrän pinta-ala</u>

Ala: $A = \pi r^2$

Yksikköympyrä: $x^2 + y^2 = 1$

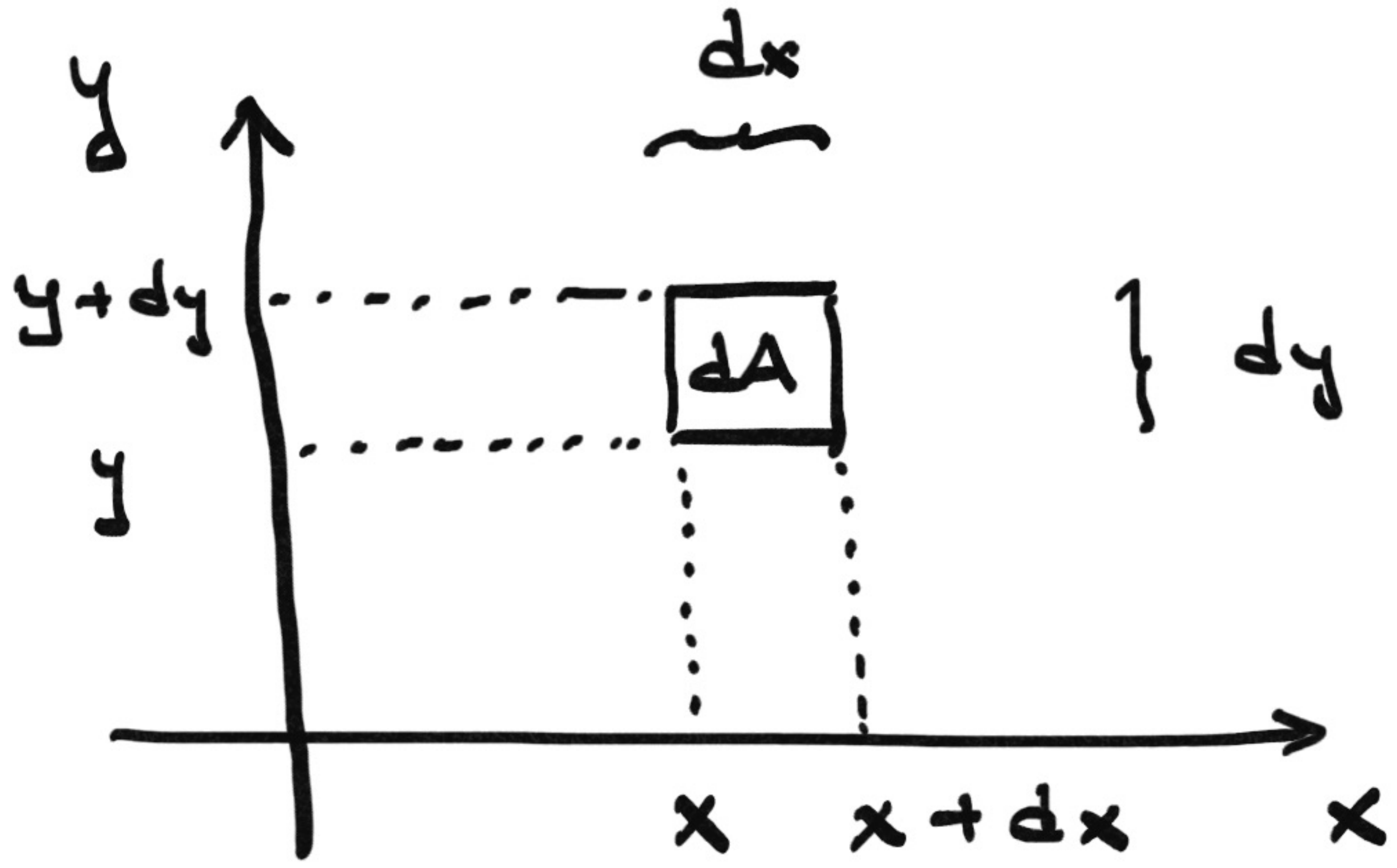
Integroimisalue

$$D = \{ (x,y) \mid x^2 + y^2 \le 1 \}$$

Integraalina:

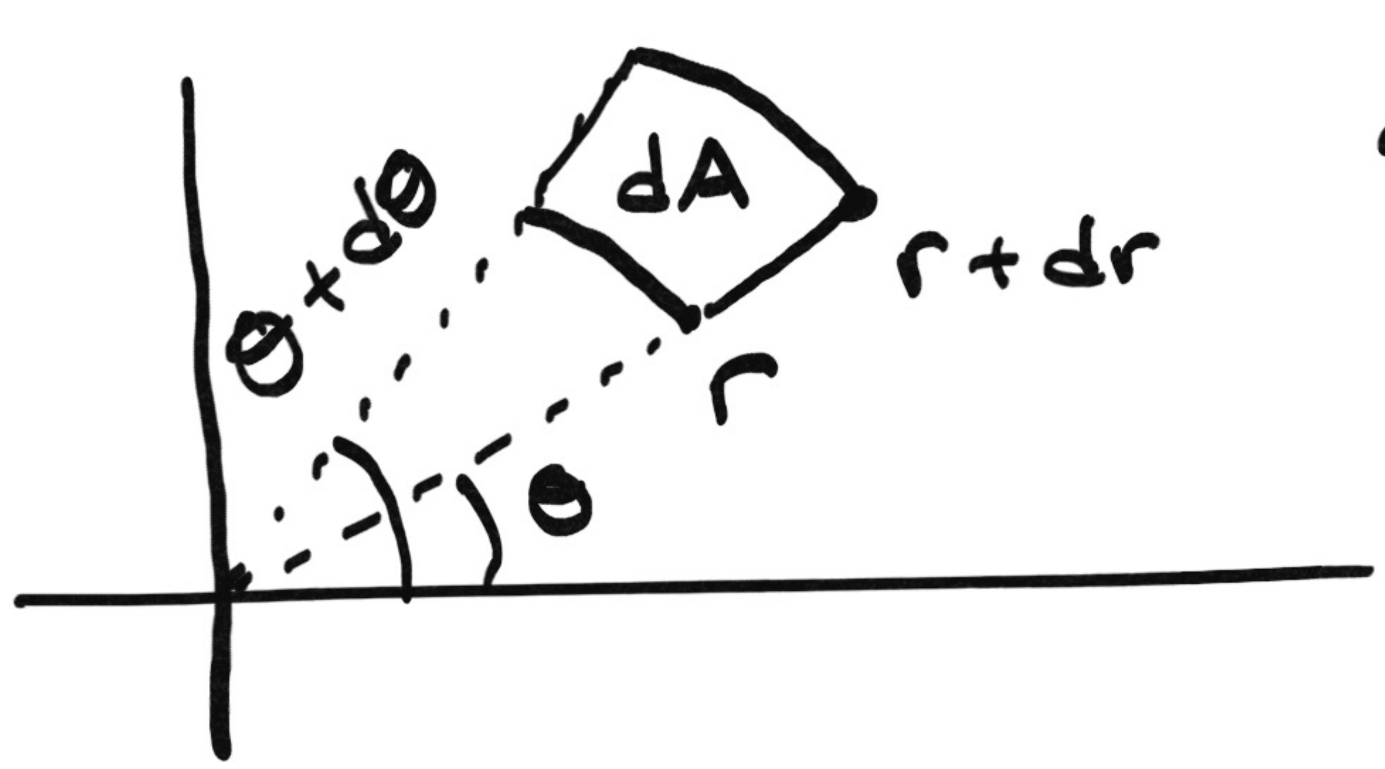
$$A = \iint_D dx\,dy$$

$$= \iint_G dA$$



differentiaali: $dA = dx\,dy$

Napakoordinaatisto: r, θ



$$dA = r\,dr\,d\theta$$

Vastinalueille:

$$dx\,dy = dA = r\,dr\,d\theta$$

Ala:

$$A = \iint_D dx\,dy = \iint_G r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} r\,d\theta\,dr$$

$$= \int_0^1 2\pi r\,dr = \pi$$

Integroimisalue G on suorakulmio napakoordinaateissa.

Yleinen tapaus

$F : G \to D$

$(x,y) \in D$

$(u,v) \in G \quad ; \quad x = x(u,v)$

$\qquad\qquad\qquad\quad\; y = y(u,v)$

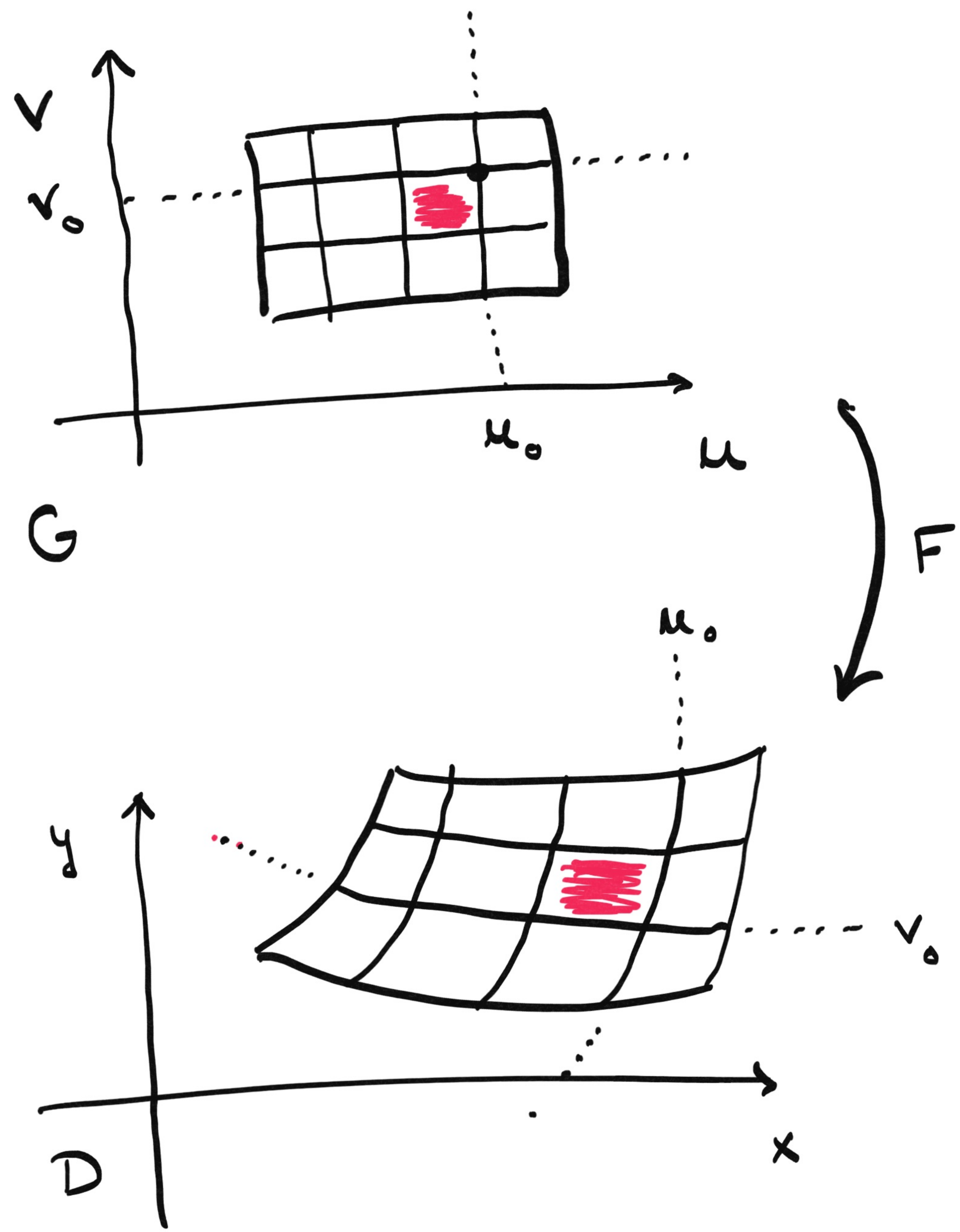
Oletukset: F

a) osittaisderivaatat olemassa ja jatkuvia

b) bijektio

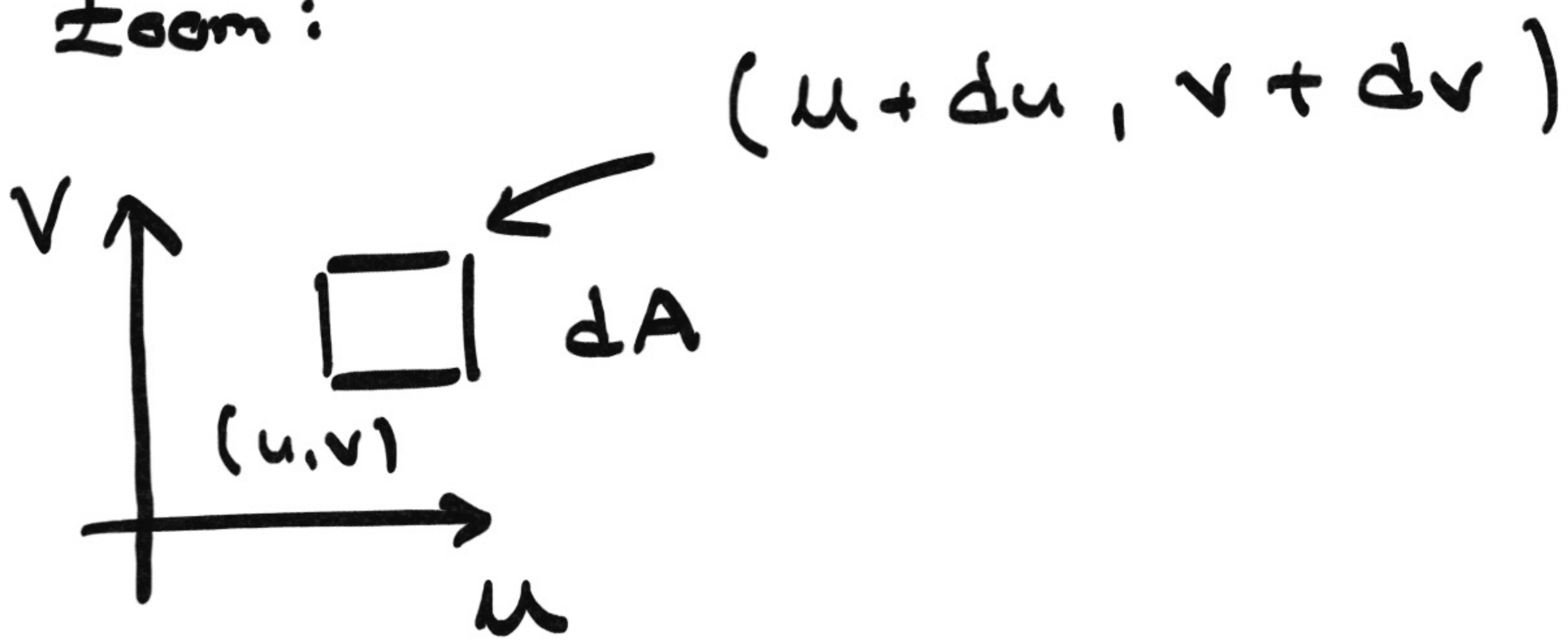
$F(u,v) = (x,y)$

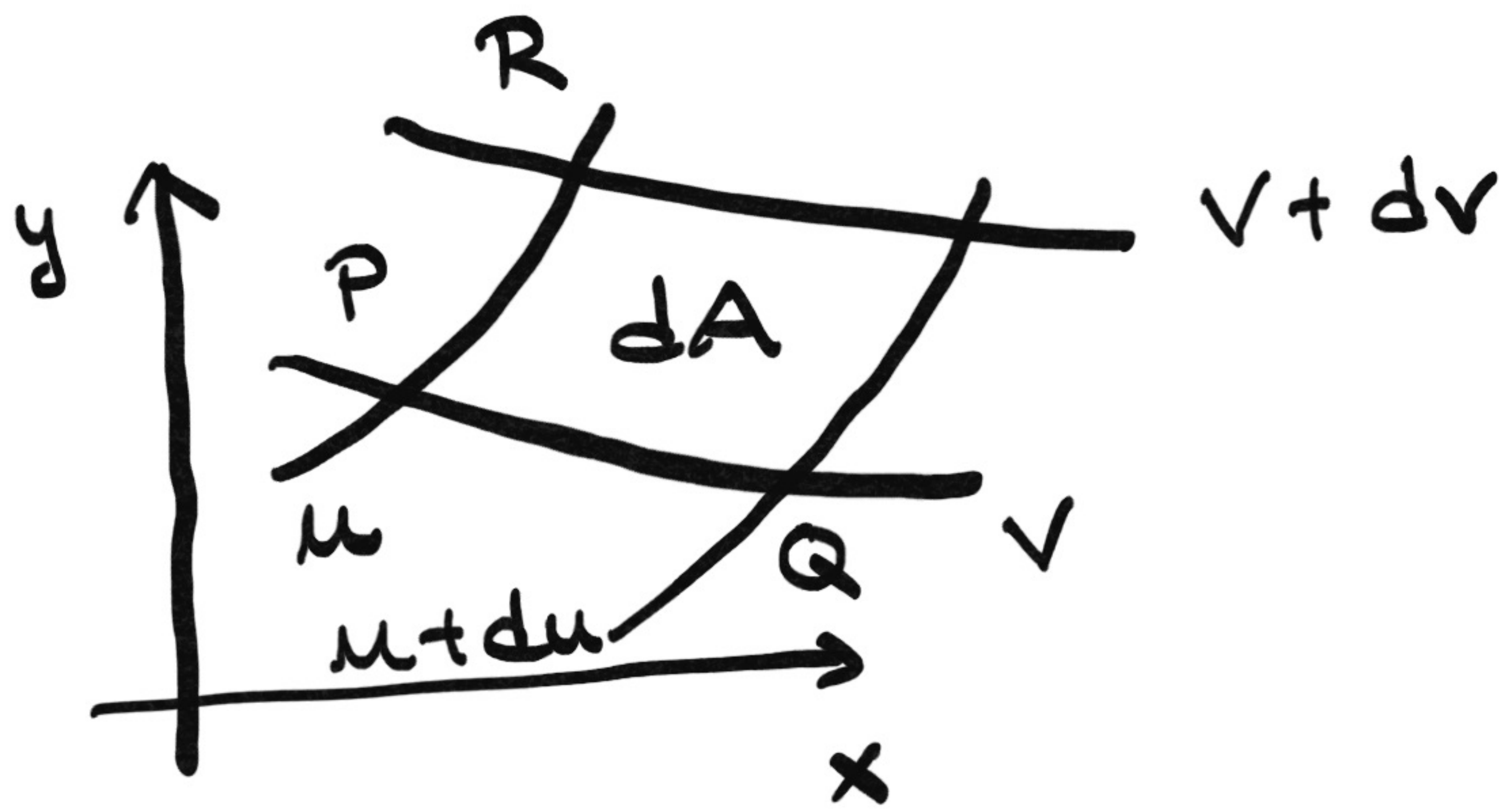
Erityisesti $D = F(G)$



$$\iint_D f(x,y)\,dx\,dy = \iint_G ?\;du\,dv$$

Zoom:





$$dA = |\overline{PQ} \times \overline{PR}|$$

$$\overline{PQ} = dx\,\underline{i} + dy\,\underline{j}$$

Nyt:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$\overline{PQ}$: v on vakio

$$\Rightarrow dv = 0$$

$$\overline{PQ} = \frac{\partial x}{\partial u} du\,\underline{i} + \frac{\partial y}{\partial u} du\,\underline{j}$$

$\overline{PR}$: u on vakio $\Rightarrow du = 0$

$$\overline{PR} = \frac{\partial x}{\partial v} dv\,\underline{i} + \frac{\partial y}{\partial v} dv\,\underline{j}$$

Eli

$$dA = \left| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} \right|$$

$$= \left| \operatorname{Det} J_{F(u,v)} \right| du\,dv$$

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Yleinen integraali on siis :

$$\iint_D f(x,y)\,dx\,dy =$$

$$\iint_G g(u,v)\,\left|\mathrm{Det}\, J_{F(u,v)}\right|\,du\,dv$$

$$g(u,v) = f(x(u,v),\, y(u,v))$$

Esimerkki Ympyrän pinta-ala

$$\iint_D dx\,dy = \iint_G \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} ; \quad \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

Esimerkki

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} = I$$

$$\hat{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = I^2$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= -\pi \lim_{R\to\infty} \int_0^R (-2r)\, e^{-r^2}\,dr$$

$$= -\pi \lim_{R\to\infty} \left(e^{-R^2} - 1\right)$$

$$= \pi$$

$$\Rightarrow I = \sqrt{\pi}$$