MUUTTUJAN VAIHTO ITEROIDUSSA INTEGRAALISSA

Sijoitusmenettely :

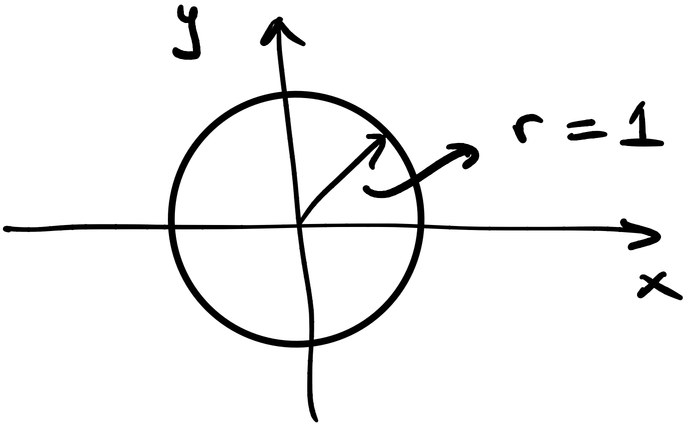
$$I = \int_a^b f(x)\,dx \; ; \qquad x = g(u) \; ; \; \underline{dx = g'(u)\,du}$$

$$a = g(\alpha) \; ; \; b = g(\beta)$$

$$I = \int_\alpha^\beta f(g(u))\, g'(u)\,du$$

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\underline{\underline{\Delta x_i}}$$

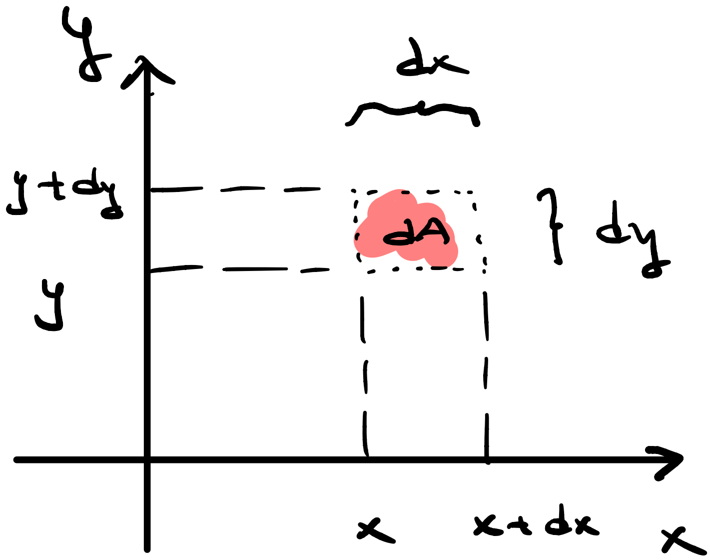
<u>Ympyrän pinta-ala</u> : $A = \pi r^2$



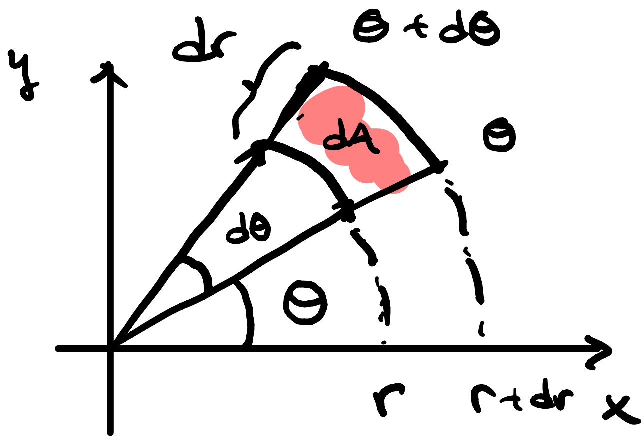
$$x^2 + y^2 = 1$$

$$D = \{ (x,y) \mid x^2 + y^2 \le 1 \}$$

$$A = \iint_D dx\,dy = \iint_G dA$$



$$dA = dx\,dy$$



$$dA = r\,d\theta\,dr$$

Kuvasta: $dx\,dy = dA = r\,d\theta\,dr$

Ala: $A = \iint_D dx\,dy = \iint_D r\,d\theta\,dr$

$$= \int_0^1 \int_0^{2\pi} r\,d\theta\,dr = \pi$$

$\hookrightarrow$ suorakulmio (?)

$= G$

Yleinen tapaus:

$\underline{F} : G \to D$; $(x,y) \in D$, $(u,v) \in G$

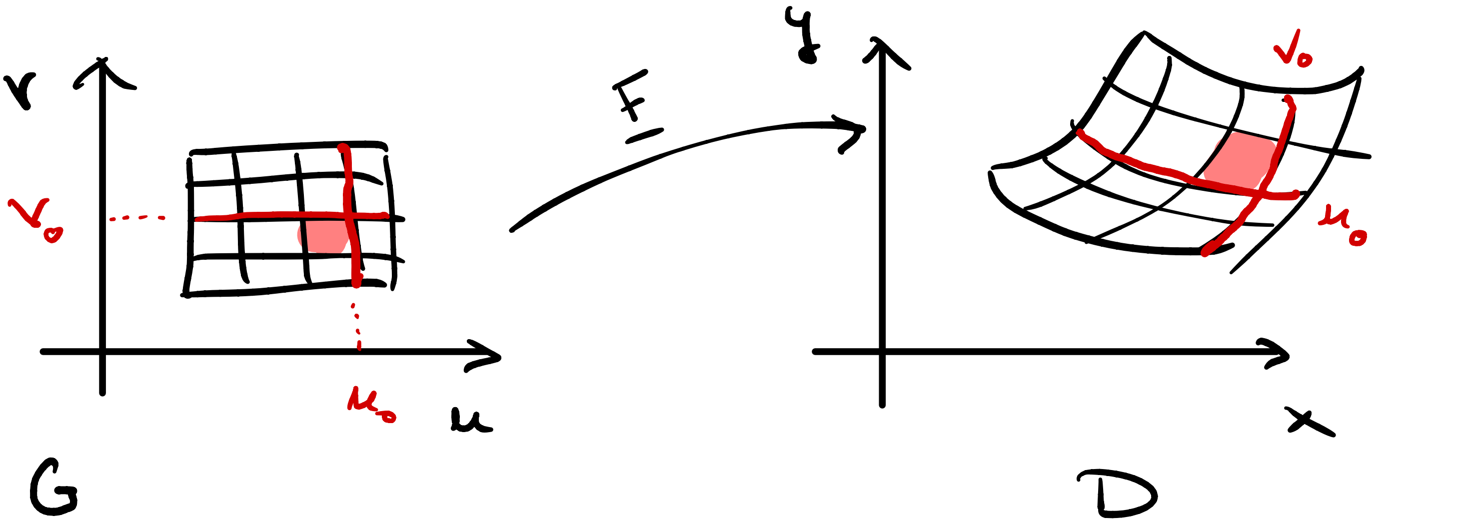
Oletukset: $\underline{F}$: a) osittaisderivaatat olemassa ja jatkuvia

b) bijektio

Jokaista (x,y) vastaa yksikäsitteinen piste (u,v) s.e.

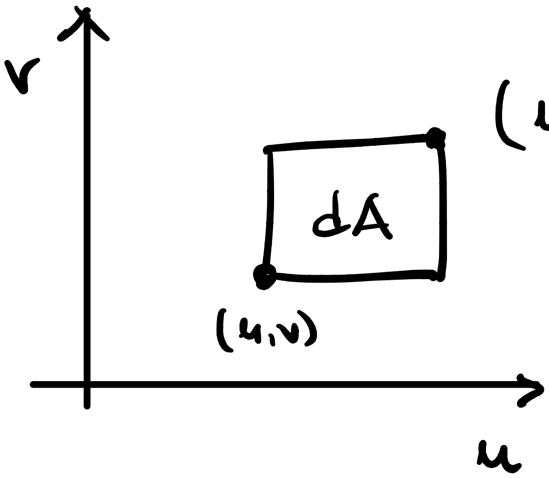
$$\underline{F}(u,v) = (x,y)$$

Tällöin $D = \underline{F}(G)$



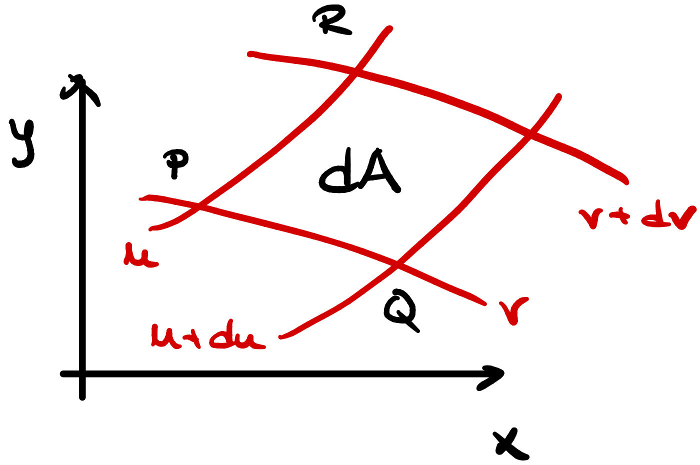
Parametrisointi: $(x(u,v), y(u,v)) = \underline{F}(u,v)$

$$\iint_D f(x,y)\,dx\,dy = \iint_G * \,du\,dv$$



$(u+du\,,\ v+dv)$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$



$$dA = \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|$$

$$\overline{PQ} = dx\,\underline{i} + dy\,\underline{j}$$

Nyt: $\quad dx = \dfrac{\partial x}{\partial u}du + \dfrac{\partial x}{\partial v}dv$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$\overline{PQ}$: v on vakio $\Rightarrow dv = 0$

$$\overline{PQ} = \frac{\partial x}{\partial u}du\,\underline{i} + \frac{\partial y}{\partial u}du\,\underline{j}$$

Vastaavasti,

$\overrightarrow{PR}$: u on vakio $\Rightarrow du = 0$

$$\overline{PR} = \frac{\partial x}{\partial v} dv \,\underline{i} + \frac{\partial y}{\partial v} dv \,\underline{j}$$

Siten,

$$dA = \left| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} \right|$$

$$= \left| \operatorname{Det} J_{\underline{F}(u,v)} \right| du\, dv$$

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

On siis saatu:

$$\iint_D f(x,y)\, dx\, dy = \iint_G g(u,v) \left| \operatorname{Det} J_{\underline{F}(u,v)} \right| du\, dv$$

$$g(u,v) = f(x(u,v), y(u,v))$$

<u>Esimerkki</u> Ympyrän pinta-ala eli muunnos napakoordinaatteihin:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

Huom! Jos kuvaus on luonnollinen suunnassa $D \to G$, niin käänteiskuvauksen determinantti on yksinkertaisesti käänteisluku.

<u>Esimerkki</u>

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2$$

$$I = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = 2\pi\int_0^{\infty} r e^{-r^2}\,dr$$

$$= -\pi \lim_{R\to\infty} \int_0^{R} (-2r)\,e^{-r^2}\,dr$$

$$\int_0^R (-2r)\, e^{-r^2} dr = e^{-R^2} - 1$$

$R \to \infty$

$$= -\pi \cdot -1 = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Kaksi erikoistapausta:

1) Lieriö- eli sylinterikoordinaatit

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow dx\,dy\,dz = r\,dr\,d\theta\,dz$$

2) Pallokoordinaatit

$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases} \Rightarrow dx\,dy\,dz =$$

$$r^2 \sin\phi\, dr\, d\phi\, d\theta$$