ELEC-E8116 Model-based control systems
Intermediate exam 2. 8.12.2022 / Solutions

- *Write the name of the course, your name and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *No literature is allowed. A calculator can be used as a calculating aid. However, it must not be used for advanced calculations, e.g. matrix calculus, Laplace transformations, connection to the Internet etc.*
- *Your solutions must be presented so that it becomes clear how you have solved the problem.*

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1. Consider a SISO-process with the transfer function

$$G(s) = \frac{s + \frac{1}{T_1}}{\left(s + \frac{1}{T_2}\right)\left(s + \frac{1}{T_3}\right)}, \quad T_1 < 0, \quad T_2 < 0, \quad T_3 > 0$$

Explain, what kind of fundamental limitations on control performance can be stated for this system? (Present also some calculations; the formulas in the end of the problems can be of help.) (5 p)

Solution:

See lecture slides, Chapter 4. The process has a RHP zero at $\mathrm{z} = -1/\mathrm{T}_1$. According to the interpolation constraint 1 it holds

$|W_S(z)| \le 1 \quad \Rightarrow \quad \omega_0 \le \left(1 - 1/S_0\right) z$ where ω_0 is the 0 dB angular frequency of the sensitivity function (under a certain parameterization of the user given frequency function $1/W_S$). In case of maximum sensitivity S_0= 2 we get $\omega_0 \le \frac{\mathrm{z}}{2} = -\frac{1}{2\mathrm{T}_1}$, but even with S_0 approaching infinity $\omega_0 \le \mathrm{z} = -\frac{1}{\mathrm{T}_1}$. The parameter ω_0 can be regarded as *bandwidth*, which means that the non-minimum phase zero gives an upper limit for bandwidth, due to the RHP zero. This bandwidth is an upper limit for achievable bandwidth of the closed loop irrespective of the controller used.

The process is unstable because of a RHP pole at $\mathrm{p}_1 = -1/\mathrm{T}_2$. According to the interpolation constraint 2 it holds

$$|W_T(p_1)| \le 1 \quad \Rightarrow \quad \omega_0 \ge \frac{p_1}{1 - 1/T_0}$$

where ω_0 is the 0 dB angular frequency of the complementary sensitivity function (under a certain parameterization of the user given frequency function $1/W_T$). Again the parameter ω_0 can be seen as a bandwidth of the closed loop system. If the maximum of T is $T_0 = 2$, we get a lower limit for the bandwidth needed to stabilize the process as $\omega_0 \geq 2\mathrm{p}_1$. Even if T_0 approaches infinity it would hold $\omega_0 \geq \mathrm{p}_1$. So the unstable pole gives rise to a minimum bandwidth needed to stabilize the process by negative feedback.

The pole $\mathrm{p} = -1/\mathrm{T}_3$ is on the LHP, so it is stable. That causes no restrictions.

It is also worth mentioning the "waterbed formula"

$$\int_0^\infty \log\ |S(i\omega)|\, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\ (\mathrm{p_i})$$

which gives a fundamental restriction to the achievable sensitivity function. The p_i :s are the RHP poles of the loop gain. If the controller has no RHP poles (usually it does not have) then here $\mathrm{Re}\ \mathrm{p_i} = \mathrm{Re}\ \mathrm{p}_2 = -1/\mathrm{T}_2$.

2. Consider the first order process

$$\begin{aligned} &\dot{x}(t) = u(t), \quad x(0) = x_0 \\ &y(t) = x(t) \end{aligned}$$

so that the state is directly measurable . It is desired to find a control that minimizes the criterion

$$J = \frac{1}{2}\int_0^\infty \left(x^2 + Ru^2\right)dt \qquad (R\ > 0)$$

Determine the optimal control law and the closed loop state equation. What is the optimal state trajectory and optimal cost? (2+1+1+1 p)

Solution:

Looking at the solution formulas we can determine that here *A=0, B=1, C=1, Q=1, R=R.* The optimization horizon is infinite so that we can use the stationary Riccati equation ($\dot{\mathrm{S}} = 0$).

$$-\dot{\mathrm{S}}(\mathrm{t}) = \mathrm{A}^{\mathrm{T}}\mathrm{S}(\mathrm{t}) + \mathrm{S}(\mathrm{t})\mathrm{A} - \mathrm{S}(\mathrm{t})\mathrm{B}\mathrm{R}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{S}(\mathrm{t}) + \mathrm{Q} = 0$$

$$-\mathrm{S}\cdot 1\cdot \frac{1}{\mathrm{R}}\cdot 1\cdot \mathrm{S} + 1 = -\frac{\mathrm{S}^2}{\mathrm{R}} + 1 = 0$$

$$\Rightarrow \mathrm{S} = \pm\sqrt{\mathrm{R}} = \sqrt{\mathrm{R}} \quad \text{(positive solution has to be chosen)}$$

The optimal control law becomes

$$u^*(t) = -Kx(t) = -R^{-1}B^T Sx(t) = -\frac{1}{R}\cdot 1\cdot\sqrt{R}\cdot x(t) = -\frac{1}{\sqrt{R}}x(t)$$

To determine the optimal trajectory note that the solution of the differential equation

$$\dot{x}(t) = ax(t), \quad x(0) = x_0$$

is

$$x(t) = x_0 e^{at}$$

[By inspection or by the Laplace transformation

$$sX(s) - x_0 = aX(s) \Rightarrow X(s) = \frac{1}{s-a}x_0 \Rightarrow x(t) = e^{at}x_0\,]$$

Then from $\dot{x}^*(t) = u^*(t) = -\frac{1}{\sqrt{R}}x^*(t)$, $x^*(t) = e^{-\frac{1}{\sqrt{R}}t}x_0$ is the optimal trajectory. The optimal cost is obtained directly from the solution formulas $J^*(0) = \frac{1}{2}x_0^2\sqrt{R}$. It could also be obtained by substituting $x^*(t)$ and $u^*(t)$ in the cost function and integrating.

3. Let us assume that you have to develop a controller for a linear multivariable process with *n* inputs and *n* outputs. Answer and explain the following:

- Define Relative Gain Array (RGA) and explain how it is used in control. (2 p)
- What is the difference between *decoupled* and *centralized* controllers? (1 p)

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- How would you design a Singular Value Decomposition (SVD)-based decoupled controller for the process? The decoupling is here considered necessary only at the zero frequency. Present the idea and also the necessary formulas. (2 p)

Solution:

For a MIMO process *G* with *n* inputs and *n* outputs the Relative Gain Array is defined as

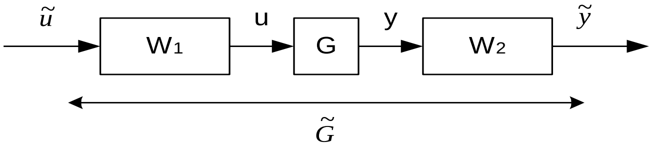
$$RGA(G) = G.\times\left(G^{-1}\right)^T$$

where .x means elementwise product of matrices (Schur or Hadamard product). RGA us used to inspect which input has the most effect to which output. In theory, the value 1 in RGA means a perfect match. RGA is normally used to to develop *decoupled* controllers, where each SISO controller $u_i - y_j$ is used independent of the *interactions* between channels.

Earlier the RGA was used only in the stationary case $G(0)$, the zero frequency. Nowadays it is understood that it is valid in other frequencies also, for example up to the gain crossover frequency ω_c. (MIMO: $\bar{\sigma}(\mathrm{G}(\mathrm{j}\omega_c)) = 0\,\mathrm{dB}$).

A centralized controller is a true multivariable controller, which controls *n* inputs based on *m* (here *n*) outputs. For example the Linear Quadratic (LQ) controller is such.

To construct a decoupling controller consider the below figure of the process with decoupling pre- and post compensators (which are part of the final controller)



If the transfer function matrix $\mathrm{W}_2\mathrm{G}\mathrm{W}_1$ is well decoupled, then SISO controllers can be used to control each channel pair $\tilde{\mathrm{u}}, \tilde{\mathrm{y}}$ separately.

$$\tilde{\mathrm{u}} = \tilde{\mathrm{F}}_\mathrm{r}\tilde{\mathrm{r}} - \tilde{\mathrm{F}}_\mathrm{y}\tilde{\mathrm{y}} \qquad (1)$$

$$\Rightarrow \mathrm{W}_1^{-1}\mathrm{u} = \tilde{\mathrm{F}}_\mathrm{r}\tilde{\mathrm{r}} - \tilde{\mathrm{F}}_\mathrm{y}\mathrm{W}_2\mathrm{y}$$
$$\Rightarrow \mathrm{u} = \mathrm{W}_1\tilde{\mathrm{F}}_\mathrm{r}\tilde{\mathrm{r}} - \mathrm{W}_1\tilde{\mathrm{F}}_\mathrm{y}\mathrm{W}_2\mathrm{y} \qquad (2)$$

Use the Singular Value Decomposition (SVD) of the process transfer function matrix at zero frequency

$$\mathrm{G}(0) = \mathrm{U}\Xi\mathrm{V}^*$$

where *U* and *V* are here real orthogonal (generally unitary) *n*x*n* matrices, and Ξ is a *nxn* real diagonal matrix with singular values in the main diagonal.

Take $W_1 = \mathrm{V}$ and $W_2 = U^* = U^\mathrm{T}$ and design a controller (1) for the process Ξ. Realize the final controller by (2).

Some formulas that might be useful:

$$\dot{x} = Ax + Bu, \quad t \geq t_0$$

$$J(t_0) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(x^TQx + u^TRu\right)dt$$

$$S(t_f) \geq 0, \quad Q \geq 0, \quad R > 0$$

$$-\dot{S}(t) = A^T S + SA - SBR^{-1}B^T S + Q, \quad t \le t_f, \quad \text{boundary condition } S(t_f)$$

$$K = R^{-1}B^T S$$

$$u = -Kx$$

$$J^*(t_0) = \frac{1}{2} x^T(t_0) S(t_0) x(t_0)$$

$$\int_0^\infty \log\ |S(i\omega)|\, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\ (\mathrm{p_i})$$

$$|W_T(p_1)| \le 1 \quad \Rightarrow \quad \omega_0 \ge \frac{p_1}{1 - 1/T_0}$$

$$|W_S(z)| \le 1 \quad \Rightarrow \quad \omega_0 \le \left(1 - 1/S_0\right) z$$