Energy and optimization models 4

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Contents

• Mixed integer linear programming (MILP)
  – Application
  – Formulation
    • Lay foundation for solving large scale system
  – Graphical representation
  – Solving MILP models
  – Sample energy models
MILP(application)

- Modeling non-convex problem
  - Shut-down of plant
  - Non-convex plant characteristic
  - unit commitment (UC) (for a given horizon)
  - Portfolio selection for energy technologies
- Logic constraints
  - Application in system control
  - AND, OR, NOT
MILP-encoding of logical constraints

- It is not easy to model constraints as linear constraints
  \[ X = Y \text{ AND } Z \Downarrow x = yz \text{ (non-linear is straightforward)} \]
- Logical constraints can be encoded using binary variables and linear constraints
  \[ X = Y \text{ AND } Z \Downarrow x \leq y; x \leq z; x \geq y+z-1 \]
  \[ X = Y \text{ OR } Z \Downarrow x \leq y+z; x \geq y; x \geq z \]
  \[ X = \text{NOT } Y \Downarrow x = 1-y \]
- Arbitrarily complex logical constraints can be formed using the above rules
  \[ Y = (Y1 \text{ AND NOT } Y2) \text{ OR } Y3 \]
  \[ \leftrightarrow Y = \text{TMP OR } Y3; \text{TMP = Y1 AND NOT } Y2 \]
  \[ \Downarrow y \leq \text{tmp+y3}; y \geq \text{tmp}; y \geq y3; \text{tmp} \leq Y1; \text{tmp} \leq 1-y2; \]
  \[ \text{tmp} \geq y1+(1-y2)-1 \]
Non-convex optimization problems

- When the objective function or some of the constraints are not convex, then the problem is non-convex.
- A non-convex problem may have several local optima.
  - It is in the general case not possible to know which of the local optima is the global optimum.
  - Necessary to explore them all.
  - It can be difficult to ensure that all local optima have been explored.
Mixed integer linear programming (MILP) model

- A mixed integer linear programming problem is similar to an LP model, but some of the variables have integer domain:
  \[
  \begin{align*}
  \text{min (max) } & \quad cx + dy \\
  \text{s.t. } & \quad Ax + By \leq b \\
  & \quad x \geq 0 \\
  & \quad y_i \in \{0, 1\} \text{ (or some other finite range of integers)}
  \end{align*}
  \]
Properties of MILP models

• Special case of non-convex problems
  – The optimum is always at a corner point of an LP model that is obtained by fixing the integer variables to some feasible values

• Reliable (but not so efficient) solution algorithms exist
  – The Branch&Bound algorithm will enumerate explicitly or implicitly the different value combinations of integer variables
  – This reduces the MILP problem into multiple LP problems

• Non-convex problems can be approximated by MILP models with arbitrarily good accuracy
  – However, the resulting model may become large and very slow to solve
Modeling shut-down of plant

- Biofuel power plant that can be shut down

\[
\begin{align*}
\text{Max } c_{\text{el}}x_{\text{el}} - c_{\text{bio}}x_{\text{bio}}; & \quad \text{ // revenue from power minus fuel cost} \\
x_{\text{el}} + P_{\text{Loss}}y &= R_{\text{bio}}x_{\text{bio}}; & \quad \text{ // constant loss in characteristic} \\
x_{\text{el}}^{\text{Min}}y \leq x_{\text{el}} \leq x_{\text{el}}^{\text{Max}}y; & \quad \text{ // bounds for power production} \\
y \in \{0,1\}; & \quad \text{ // y indicates if plant is in operation}
\end{align*}
\]

- The binary y-variable acts as a switch to determine if the plant is on (y=1) or off (y=0)
- The y-variable affects both the plant characteristic and bounds for power output
MILP-encoding of non-convex problems

- A non-convex optimization problem is of form
  \[ \text{Min } f(x); \text{ s.t. } x \in X \]
  - where \( f() \) is a non-convex function, or
  - \( X \) is a non-convex set (or both)
- The non-convex set can be partitioned into convex subsets
  \( X = \bigcup X_i \)
- Each subset is represented approximately by linear inequalities
  \[ A_i x \leq b_i + M(1-y_i) \]
  - \( M \) is a big number that deactivates the constraints of subset \( X_i \)
  using binary variables
  \[ \Sigma y_i = 1 \]
  \[ y_i \in \{0,1\} \]

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MILP-encoding of non-convex characteristic

\[ f(x) \]

\[ x_{n_0}, x_{n_{i-1}}, x_{n_i}, x_{n_r} \]

\[ y_1, y_i, y_r, \lambda^1_{n_0}, \lambda^i_{n_{i-1}}, \lambda^n_{n_i}, \lambda^r_{n_r} \]

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Convex characteristic modeling

- Convex characteristic can be encoded as an LP model
  - Any point \((x_1, x_2, \ldots, x_k)\) \((k \geq 2)\) in the k-dimensional space can be represented by the convex combination of selected (known) points as

  \[
  (x_1, x_2, \ldots, x_k) = \left( \sum_{i=1}^{n} \alpha_i x_1^i, \sum_{i=1}^{n} \alpha_i x_2^i, \ldots, \sum_{i=1}^{n} \alpha_i x_k^i \right),
  \]

  \[
  \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0, i = 1, \ldots, n
  \]

  \[
  (x_1^i, x_2^i, \ldots, x_k^i), i = 1, \ldots, n \text{ are } n \text{ selected (known) points.}
  \]

  - Convex combination forms a region and minimization can be used to model the lower envelop of the region

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Convex combination forming a region

• Take points of two-dimensional plane as example
  
  – \((x, y) = \alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) + \ldots + \alpha_n(x_n, y_n) = (x_1 \alpha_1 + x_2 \alpha_2 + \ldots + x_n \alpha_n, y_1 \alpha_1 + y_2 \alpha_2 + \ldots + y_n \alpha_n), \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1, \alpha_i \geq 0, i=1,\ldots,n\)
  
  – \((20,40), (50,100), (35,10) (n=2, n=3)\)

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Convex power-only plant

- The power plant cost characteristic defines in the y-x plane as a function $c=f(p)$, $p$ is power production.
- We encode the model as a convex combination of extreme (corner) points $(p_j, c_j)$

$$\min \sum_j c_j x_j$$

s.t.

$$\sum_j p_j x_j = p \quad // \quad \text{variable power prod.}$$

$$\sum_j x_j = 1 \quad // \quad \text{convex comb.}$$

$$x_j \geq 0$$
Feasible operating region of the power plant characteristic defines in the P-Q plane

\[ p = \text{power production}, \quad q = \text{heat production}, \quad c = \text{operating cost} \]

We encode the model as a convex combination of extreme (corner) points

\[ \min \sum c_j x_j \]

s.t.
\[ \sum p_j x_j = p \quad // \text{variable power prod.} \]
\[ \sum q_j x_j = q \quad // \text{fixed heat demand} \]
\[ \sum x_j = 1 \quad // \text{convex comb.} \]
\[ x_j \geq 0 \]
MILP-encoding of non-convex piecewise linear function
(line segment approach)

- The non-convex objective function can be approximated by a piecewise linear function
- E.g. 1-dimensional function \( f = f(x) = -x^2 \)
  - Choose points \((x_i, f_i)\) and
  - define \((x,f)\) as convex combination of \(r\) linear segments using
    - non-negative continuous variables \(\lambda_i\) \((i=0,...,r)\) and
    - binary variables \(y_i\) \((i=1,...,r)\) to activate one segment at time
  - Formulation is given below
For a given $x$, the line segment approach determines which line segment and corresponding endpoints are active. The idea of the line segment approach is to model how to disable an endpoint. If we know which points are disabled, then it is easy for us to know which points are active. According to the figure, Points $(x_{i-1}, f_{i-1})$ and $(x_i, f_i)$ are two endpoint of line segment $i$. For modeling, each line segment $i$ is associated with a binary variable $y_i$ and each endpoint $(x_i, f_i)$ is associated with a nonnegative value $\lambda_i$. $\lambda_i=0$ means that point $(x_i, f_i)$ is not active (or is disabled) and $y_i=0$ means that line segment $i$ is not active.
From the mathematical viewpoint, $\lambda_i=0$ can be represented as $\lambda_i \geq 0$ and $\lambda_i \leq 0$. Since $\lambda_i \geq 0$ is given, to disable point $(x_i, f_i)$, we need to find the inequality relation making $\lambda_i \leq 0$. Point $(x_i, f_i)$ can be associated with at most two line segments and Point $(x_i, f_i)$ is disabled if all the line segment(s) associated with it are disabled. For the figure, Point $(x_0, f_0)$ is associated with line segment 1 and Point $(x_r, f_r)$ with line segment.
MILP-encoding of non-convex characteristic (line segment approach)

\[
\begin{align*}
\min f(x) &= \sum_{i=0}^{r} \lambda_i f_i \quad (M1_1) \\
\text{Subject to} \\
x &= \sum_{i=0}^{r} \lambda_i x_i \quad (M1_2) \\
\sum_{i=0}^{r} \lambda_i &= 1 \quad (M1_3) \\
\sum_{i=1}^{r} y_i &= 1 \quad (M1_4) \\
\lambda_0 &\leq y_1 \quad (M1_5) \\
\lambda_i &\leq y_i + y_{i+1}, i = 1, ..., r - 1 \quad (M1_6) \\
\lambda_r &\leq y_r \quad (M1_7) \\
\lambda_i &\geq 0, i = 0, ..., r \quad (M1_8) \\
y_i &\in \{0, 1\}, i = 1, ..., r \quad (M1_9)
\end{align*}
\]

Objective \((M1_1)\), constraints \((M1_2), (M1_3)\) and \((M1_8)\) form convex combination for given points. Constraints \((M1_4)\) means that only one segment can be active at one time. Constraints \((M1_5) - (M1_7)\) mean that points will be disabled if all the segments associated with it are not active.
MILP-encoding of a generic piecewise linear function (convex partition)

- In some cases, for the one variable case, it is more efficient to model a generic (non-convex) piecewise linear function according to convex partition than the previous line segment modeling if some convex areas contain more than one line segment because fewer binary variables are needed. One convex area is associated with one binary variable.

- Modeling a convex piecewise linear function or any convex function does not need to introduce any binary variable
A non-convex characteristic can be divided into $r$ convex regions.

Let $\left(x^I_{n-1}, f^I_{n-1}\right), (i = 2, ..., r)$ are delimiting points between two convex regions $i-1$ and $i$ and associated with both regions. Point $\left(x^1_{n_0}, f^1_{n_0}\right)$ is only associated with convex region 1. Point $\left(x^r_{n_r}, f^r_{n_r}\right)$ is only associated with convex region $r$. Points $\left(x^I_j, f^I_j\right), i = 1, ..., r, j = n_{i-1} + 1, ..., n_i - 1$ are associated with convex region $i$. Let

$\lambda^I_n$ is associated with $\left(x^I_n, f^I_n\right)$, $\lambda^I_j (i = 1, ..., r, j = n_{i-1}, ..., n_i - 1)$ are associated with $\left(x^I_j, f^I_j\right)$. Binary variable $y_i (i = 1, ..., r)$ are associated with convex region $i$. 

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MILP-encoding of a generic piecewise linear function (convex partition)

\[ \min f(x) = \sum_{i=1}^{r} \sum_{j=n_{i-1}}^{n_{i}-1} \lambda_j^i f_j + \lambda_{n_i} f_{n_r} \]  
(M2_1)

subject to

\[ x = \sum_{i=1}^{r} \sum_{j=n_{i-1}}^{n_{i}-1} \lambda_j^i x_j + \lambda_{n_i} x_{n_r} , \]  
(M2_2)

\[ \sum_{i=1}^{r} \sum_{j=n_{i-1}}^{n_{i}-1} \lambda_j^i + \lambda_{n_i} = 1 , \]  
(M2_3)

\[ \sum_{i=1}^{r} y_i = 1 , \]  
(M2_4)

\[ \sum_{j=n_{i-1}}^{n_{i}-1} \lambda_j^i + \lambda_{n_i}^{i+1} = y_i , i = 1, \ldots , r - 1 , \]  
(M2_5)

\[ \sum_{j=n_{r-1}}^{n_r} \lambda_j = y_r , \]  
(M2_6)

\[ \lambda_{n_i}, \lambda_j^i \geq 0 , i = 1, \ldots , r , j = n_{i-1}, \ldots , n_i - 1 , \]  
(M2_7)

\[ y_i \in \{0,1\} , i = 1, \ldots , r . \]  
(M2_8)

Objective (M2_1) and constraints M2_2, M2_3 and (M2_7) form the convex combination of given points. Constraints (M2_4) means only area is active at one time. Constraints (M2_5) and (M2_6) form convex combination in each area when \( y_i=1(i=1,\ldots,r) \) and disable all the points in each area when \( y_i=0. \)

The common constraints for two approaches are formula for objective function and the first three constraints. The ways to associate \( \lambda \) and \( y \) are different.
For a given $x$, the convex partition approach first determines how many convex regions the non-convex function contains. Then it determines which convex region and corresponding points are active. It is the direct extension for the convex combination approach for convex characteristic modeling. For modeling, each convex region $i$ is associated with a binary variable $y_i$ and each point $(x_k, f_k)$ is associated with a nonnegative value $\lambda_k$. When $y_i=1$, it needs to determine which points $(x_k, f_k)$ in the convex region $i$ is active (i.e. the value of $\lambda_k$). When $y_i=0$, all the points in convex region $i$ will be disabled (all $\lambda_k$ associated with the point $(x_k, f_k)$ will be zero).
We use the same figure above but the similar notation as the line segment approach. The non-convex characteristic is divided into three convex regions. Region 1 consists of points 0, 1 and 2, region 2 consists of points 2, 3 and 4, and region 3 consists of points 4, 5 and 6. Region $i$ is associated with binary variable $y_i$, and $y_i=1$ means that region $i$ is active. Each time, only one region can be active. $(x_i, f_i)$ $(i=0,...,6)$ are given points in the two dimensional space. Point $i$ is associated with non-negative variable $\lambda_i$. 

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Example

\[
\begin{align*}
\text{min } f(x) &= \sum_{i=0}^{6} f_i \lambda_i \\
\text{subject to} & \\
x &= \sum_{i=0}^{6} x_i \lambda_i \\
\sum_{i=0}^{6} \lambda_i &= 1 \\
\sum_{i=1}^{3} y_i &= 1, \\
\sum_{i=0}^{2} \lambda_i &= y_1 \\
\sum_{i=2}^{4} \lambda_i &= y_2 \\
\sum_{i=4}^{6} \lambda_i &= y_3 \\
\lambda_i &\geq 0, i = 0 \ldots 6, \\
y_i &\in \{0,1\}, i = 1,2,3.
\end{align*}
\]

Objective (ME2_1), constraint2 (ME2_2), (ME2_3) and Constraints (ME2_8) form the convex combination of given points. Constraints (ME2_4) means that only one convex area can be active. Constraints (ME2_5)-(ME2_7) form the convex combination of each area when the corresponding y-variable is one and disable all the points in the area if corresponding y-variable is zero.
Non-convex CHP model

• Necessary when either (or both)
  – The cost function is non-convex
  – P-Q the characteristic is non-convex
    • E.g. when it is necessary to optimize the shut-down of the plant

• Idea (convex partition technique)
  – Partition objective function into convex parts
  – Partition characteristic into convex parts
  – Use 0/1 variables to choose in which area to operate
Sample non-convex cogeneration model

Allocation of characteristic points to convex sub-areas

<table>
<thead>
<tr>
<th>Area</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
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Non-convex cogeneration model

- Characteristic is partitioned in three convex parts
- \( A_j \) is set of areas to which \( x_j \) belongs
- Define zero-one variables \( y_1, y_2, y_3 \), and allow exactly one of them to have value 1.
- \( y \)-variables select which corner points are allowed in the convex combination

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\[
x_j \leq \sum_{a \in A_j} y_a, \quad j \in J_u, \quad u \in U^*,
\]

\[
\sum_{a \in A_u} y_a = 1, \quad u \in U^*,
\]

\[
y_a \in \{0, 1\}, \quad a \in A_u, \quad u \in U^*.
\]
In principle it is possible to solve MILP problems using brute force:

- Choose a value combination of integer variables
- Solve the resulting LP problem
- The best feasible solution among all combinations gives the optimum

The number of problems to solve is exponential with respect to number of variables:

- With \( N \) binary variables, there are \( 2^N \) combinations
- \( N=10 \rightarrow 1024, 20 \rightarrow 10^6, 30 \rightarrow 10^9, \ldots \)
Solving MILP models

- The Branch & Bound (BB), Branch & Cut (BC) as well dynamic programming (DP) algorithm solves MILP models more efficiently by solving only a small fraction of all combinations (implicitly enumeration)
  - Still solution time may be exponential

- Standard software
  - CPLEX, GAMS, Lindo, Lingo, Excel Solver …

- Very efficient specialized algorithms exist for the extreme point formulation
  - Power Simplex, or Extended Power Simplex, Tri-Commodity Simplex based BB.
Additional Reading


More reading on specialized algorithms

Non-convex power plant modelling in energy optimisation

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Available online 10 March 2005
More reading on specialized algorithm

An efficient envelope-based Branch and Bound algorithm for non-convex combined heat and power production planning

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Review questions

- Students should reflect lecture material and consider review questions at home before next lecture by summarizing materials.
- Students should be prepared to answer review questions at beginning of next lecture.

1. For an individual plant, it is not difficult to get the input and output of plant by measurement. Why is it necessary to model the individual plant.

2. Any line segment can be modelled by introducing a binary variable associated with it. But it is redundant for modeling convex characteristics. Give reasons for this.

3. For modeling non-convex piecewise linear function, reflect the similarity and difference for defining decision variables and for associating $\lambda$ with $y$ for line segment approach and convex partition approach. Writing constraints (M1_5) – (M1_7) in the similar way as constraints (M2_5) and (M2_6)

4. Why the convex problem is easy to solve while non-convex problem is difficult to solve.

5. List typical solution approaches for the MILP problem.

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