ELEC-E8116 Model-based control systems
Full exam 13. 12. 2022 Solutions

- *Write the name of the course, your name and student number to each answer sheet.*
- *There are five (5) problems and each one must be answered.*
- *No literature is allowed. A calculator can be used as a calculating aid. However, it must not be used for advanced calculations, e.g. matrix calculus, Laplace transformations, connection to the Internet etc.*
- *Your solutions must be presented so that it becomes clear how you have solved the problem.*

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1. Explain briefly the following concepts (max 1p each)

- Principle of Optimality
- Dynamic programming
- Waterbed effect
- Robust stability
- ”Push-through” rule
- Internal model control

Solution:

- The Principle of Optimality means that no matter how you have come to a state on an optimal route, the remaining controls to the end must be optimal starting from that state.
- Dynamic programming means algorithms that use the Principle of Optimality directly. This can mean calculating algorithms e.g. in solving scheduling problems or deriving theoretical solutions to e.g. LQ problems.
- Waterbed effect is a kind of fundamental restriction in control. It can be given as a formula, but basically it means that decreasing the sensitivity function of a system in some frequencies must lead to increase in some other frequencies.
- Robust stability means that the controller designed for a system leads to a stable closed loop, even if the process model has uncertainties (with a given bound) or the system has disturbances. In other words, the uncertainty (usually on a given frequency band) or disturbances have been taken into consideration when designing the controller.
- “Push Through” rule means the mathematical identity

$$\mathrm{A}(\mathrm{I}+\mathrm{BA})^{-1}=(\mathrm{I}+\mathrm{AB})^{-1}\mathrm{A}$$

  where *A* and *B* are matrices such that *AB* and *BA* are both defined and the inverses exist. *I’s* are identity matrices with compatible dimensions.
- Internal Model Control means a control topology, where the system model is explicitly used in the control structure. The feedback information consists of only the difference between the process output signal and the predicted output (which is calculated by the process model).

2. Consider a SISO system in a two-degrees-of-freedom control configuration. Let the loop transfer function be $L(j\omega) = G(j\omega)F_y(j\omega)$, where the symbols are standard used in the course.

a. Define the *sensitivity* and *complementary sensitivity functions* and determine where in the complex plane it holds

$$|S(j\omega)| < 1, \quad |S(j\omega| = 1, \quad |T(j\omega)| < 1 \text{ and } |T(j\omega)| = 1 \qquad \text{(3 p)}$$

b. Determine the point(s) in the complex plane where $|S(j\omega)| = |T(j\omega)| = 1$.

Hint to the problem: In the complex plane (xy) let $L(j\omega) = x(\omega) + jy(\omega)$. (3 p)

Solution:

a. Standard definitions, see lecture slides, Chapter 3. In the SISO case

$$L(j\omega) = G(j\omega)F_y(j\omega)$$
$$S(j\omega) = \frac{1}{1 + L(j\omega)}$$
$$T(j\omega) = \frac{L(j\omega)}{1 + L(j\omega)}$$

Denote $L(j\omega) = x(\omega) + jy(\omega)$ and calculate

$$S = \frac{1}{1 + x + jy} \Rightarrow |S| = \frac{1}{\sqrt{(1+x)^2 + y^2}} \Rightarrow (1+x)^2 + y^2 = \frac{1}{|S|^2}$$

In the complex (x-y) plane this is a circle with the center point (-1,0) and radius $1/|S|$. Consider the circle with radius 1. On the circle $|S| = 1$, outside the circle $|S| < 1$, inside the circle $|S| > 1$. So when the Nyquist diagram of $L(j\omega)$ enters the circle from outside to inside the absolute value of S obtains the above values accordingly.

Now $T = \dfrac{L}{1+L} = \dfrac{x + iy}{1 + x + iy} \Rightarrow |T| = \dfrac{\sqrt{x^2 + y^2}}{\sqrt{(1+x)^2 + y^2}} = \sqrt{\dfrac{x^2 + y^2}{(1+x)^2 + y^2}} = \sqrt{\dfrac{x^2 + y^2}{x^2 + y^2 + 2x + 1}}$

Clearly
$$|T| = 1 \Rightarrow 2x + 1 = 0, \Rightarrow x = -1/2$$
$$|T| < 1 \Rightarrow 2x + 1 > 0, \Rightarrow x > -1/2$$

The absolute value of T is 1 on the line x= -1/2 on the complex plane. $|T| < 1$ holds for all points to the right of this line.

b. Based on the solution of part a, it is obvious that these points are the intersection of the line x = -1/2 and the unit circle centred at (-1,0). Substituting x = -1/2 into

$$(1+x)^2 + y^2 = 1$$

gives after simple calculation $y = \pm\frac{1}{2}\sqrt{3}$. So the two points $\left(-\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \left(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$ fulfil the required condition.

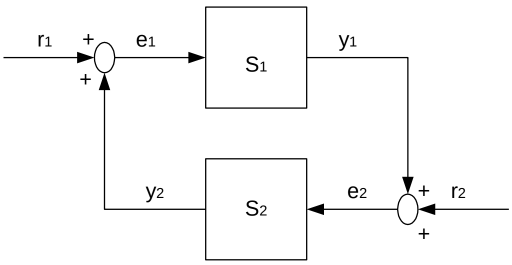
3. **a.** Explain the *Small Gain Theorem*. (2 p)

b. Explain shortly what is meant by the Relative Gain Array (RGA) and what is its meaning in control engineering. (2 p)

c. Consider a linear SISO system. Explain shortly what different definitions there exist for the concept *bandwidth*. Explain these shortly. How can they be characterized in terms of control performance? (2 p)

Solution:

a.



Small Gain Theorem": The closed loop system is BIBO stable, if the product of the system gains is smaller than1.

$$\|S_2\| \cdot \|S_1\| < 1$$

If S1 and S2 are linear, a weaker condition follows

$$\|S_1 S_2\| < 1$$

b. For a MIMO process G with n inputs and n outputs the Relative Gain Array is defined as

$$\mathrm{RGA(G)} = \mathrm{G}.\times\left(\mathrm{G}^{-1}\right)^{\mathrm{T}}$$

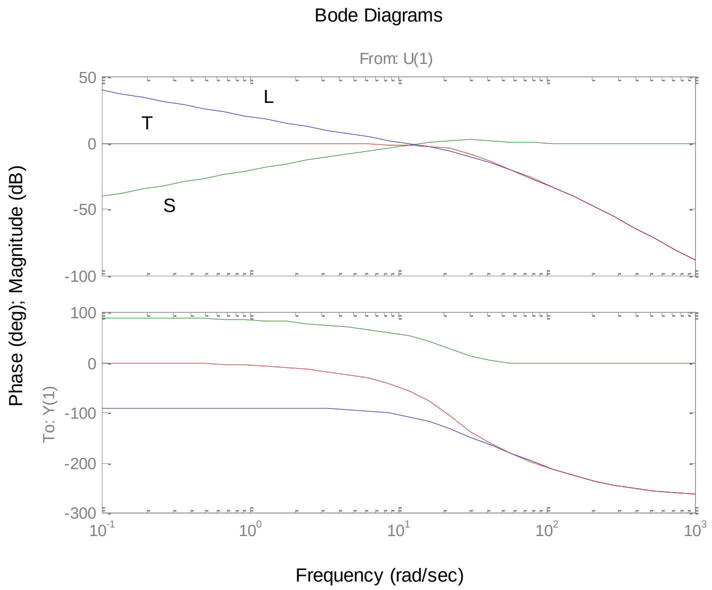
where .x means elementwise product of matrices (Schur or Hadamard product). RGA us used to inspect which input has the most effect to which output. In theory, the value

1 in RGA means a perfect match. RGA is normally used to to develop *decoupled* controllers, where each SISO controller $u_i - y_j$ is used independent of the *interactions* between channels.

Earlier the RGA was used only in the stationary case $G(0)$, the zero frequency. Nowadays it is understood that it is valid in other frequencies also, for example up to the gain crossover frequency ω_c. (MIMO: $\bar{\sigma}(G(j\omega_c)) = 0\,\mathrm{dB}$).

In control engineering the idea has been to construct decoupling SISO controllers to control MIMO processes. RGA has then been used to decide the decoupling variables, i.e. which input is used to control which output.

c. The bandwidth means generally the (angular) frequency range for which the system performs well, i.e. the output can follow the input. For linear systems the concept is easy to understand by Bode plot analysis. In control engineering the bandwidth is usually used to denote the frequency range of the sinusoidal reference signal, which can be controlled such that the output of the closed loop system can follow the reference.



There are different definitions for bandwidth. The simplest is the range $0\ldots\omega_c$ where ω_c denotes the gain crossover frequency. Other definitions: $0\ldots\,\omega_B$ where ω_B is the angular frequency at which the sensitivity function S reaches $1/\sqrt{2}$, (-3 dB) from *below*; or at which the complementary sensitivity function T reaches $1/\sqrt{2}$, (-3 dB) from *above*.

4. Write a short (max one page) description of the concept "*Fundamental restrictions in control*". (What are they, how do they affect control design, etc.) (6 p)

Solution:

Chapter 4 in the lecture slides discusses fundamental restrictions. They can be categorized as follows:

-unstable systems
-systems with delay
-non-minimum phase systems
-limitations in control signal range
-system inverse

Unstable systems and non-minimum phase systems cause restrictions in the closed loop bandwidth. Non-minimum phase systems and systems with pure delay (comparable to the case with RHP zero e.g. by using the Padé approximation) give an upper bandwidth for the closed loop, which cannot be exceeded by any controller. The RHP poles (unstable system) gives rise to a lowest bandwidth which has to be exceeded to stabilize the closed loop system.

The fundamental formula relating the sensitivity functions

$$\mathrm{S}(\mathrm{j}\omega)+\mathrm{T}(\mathrm{j}\omega)=\mathrm{I}$$

can be seen as a fundamental restriction. The "waterbed formula" limits the possibility to shape *S* in general and specifically if the open loop has RHP poles. The interpolation formulas 1 and 2 give fundamental restrictions to crossover frequencies of *S* and *T*, if there are RHP zeros or poles in the process.

Limitations in control signal range limit the possibility to full rejection of disturbances entering at specific frequencies. That is linked to the process inverse, which is implicitly involved in all control design algorithms. These issues can be analysed and formulas can be developed (see lecture slides).

TO GET full points, most (but not necessarily all) of the above items have to be mentioned.

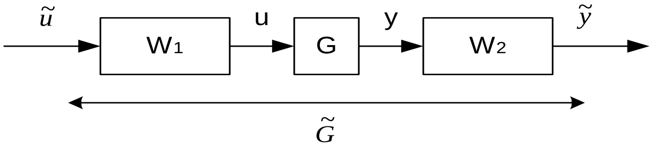
5. Consider the following multivariable system

$$G(s)=\begin{bmatrix} \dfrac{2}{s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix}$$

for which at zero frequency

$$G(0)=\begin{bmatrix} 2 & 1.5 \\ 1 & 1 \end{bmatrix} \qquad \mathrm{RGA}\big(G(0)\big)=\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

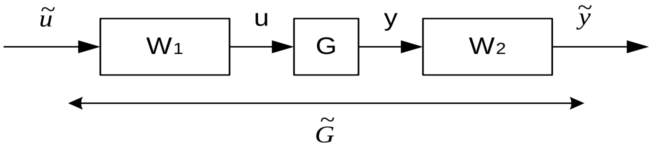
a. How would you choose the *pairing*, if two SISO loops would be used for control. (2 p)
b. How would you design a *decoupling* multivariable controller, when the decoupling is designed at the zero frequency. You do not have to design the controller numerically, just explain in detail how you would do it. (4 p)



Solution:

a. No pairing is good without decouplers. The coupling $u_1 \leftrightarrow y_1, u_2 \leftrightarrow y_2$ seems better (RGA terms positive), but not really good (RGA terms not near 1). The coupling $u_1 \leftrightarrow y_2, u_2 \leftrightarrow y_1$ seems totally unacceptable, because of negative signs in RGA components.

b. To construct a decoupling controller consider the below figure of the process with decoupling pre- and post compensators (which are part of the final controller)



If the transfer function matrix W_2GW_1 is well decoupled, then SISO controllers can be used to control each channel pair $\tilde{u},\tilde{y}$ separately.

$$\tilde{u} = \tilde{F}_r\tilde{r} - \tilde{F}_y\tilde{y} \qquad (1)$$

$$\Rightarrow W_1^{-1}u = \tilde{F}_r\tilde{r} - \tilde{F}_yW_2y$$
$$\Rightarrow u = W_1\tilde{F}_r\tilde{r} - W_1\tilde{F}_yW_2y \qquad (2)$$

Use the Singular Value Decomposition (SVD) of the process transfer function matrix at zero frequency

$$G(0) = U\Xi V^*$$

where U and V are here real orthogonal (generally unitary) nxn matrices, and Ξ is a nxn real diagonal matrix with singular values in the main diagonal.

Take $W_1 = V$ and $W_2 = U^* = U^T$ and design a controller (1) for the process Ξ. Realize the final controller by (2).