Lecture 5

Learning goals

- To know how the electromagnetic field is quantized: the concept of photons.
- To learn three important examples of quantum states of electromagnetic field: the Fock states, the coherent states, and the squeezed states. Note that also many other systems than the electromagnetic field are well described by these quantum states: the concepts are generic.

Second quantization continued

9 Quantization of the electromagnetic field

Literature: F. Schwabl, Advanced Quantum Mechanics (Springer), Chapter 14.4.; D.F. Walls and G.J. Milburn, Quantum Optics (Springer), Chapters 2.1-2.4; P. Meystre and M. Sargent III, Elements of quantum optics (Springer), Chapters 12.1, 12.4 (Also S.M. Barnett and P.M. Radmore, Methods in Theoretical Quantum Optics (Oxford University Press))

Planck 1900: distribution of thermal light that assumes quantized energy.
Einstein 1905: explanation of the photoelectric effect by assuming quantized energy. The concept of "photons"; quantum of the electromagnetic field with the energy $h\nu$.

Here we now quantize the electromagnetic field, and study its interaction with a two-level system (e.g. an electronic transition in an atom). The treatment provides an example of the use of the second quantization formalism presented in the previous lectures. Now the field is the electromagnetic field; we will have annihilation and creation operators corresponding to the quantized excitations, in this case photons; the state of the system can be expressed by applying combinations of the annihilation and creation operators to the vacuum field.

The concepts introduced in this lecture are very general and fundamental, and applied in various other contexts as well in addition to the interaction between light and matter (for instance in nanoelectronics systems, ultracold quantum gases and quantum information, as will be seen later in the course).

9.1 General procedure for quantizing a field

Consider a field $X(\mathbf{x},t)$ (fields are often also denoted $\Psi(\mathbf{x},t)$). The canonical conjugate $P(\mathbf{x},t)$ from the Lagrangian $\mathcal{L}(X, \frac{\partial X}{\partial t})$ is

$$P = \frac{\delta \mathcal{L}}{\delta \left(\frac{\partial X}{\partial t}\right)} \tag{9.1}$$

To quantize, then define:

$$[X, P] = \frac{ih}{2\pi}. \tag{9.2}$$

For electromagnetic radiation, X corresponds to the vector potential $\mathbf{A}(\mathbf{x},t)$, and P corresponds to the electric field $\mathbf{E}$, as will be discussed below. One might then think that by imposing a relationship of the type (9.2) the field could be quantized. This, however, is not correct because Equation (9.2) holds only for unconstrained dynamical variables. The vector potential $\mathbf{A}(\mathbf{x},t)$ is not unconstrained: there is freedom to choose the gauge, for instance the Coulomb gauge $\nabla \cdot \mathbf{A}(\mathbf{x}, t) = 0$, and then only two of the three components of $\mathbf{A}(\mathbf{x}, t)$ are independent. Therefore quantization of the electro-magnetic field is a bit more complicated task. Let us now proceed to calculate the canonical conjugate.

The Lagrangian density for the electromagnetic field (we assume the scalar potential is zero, i.e. there is no source, which holds in the Coulomb gauge that we use throughout these lecture notes):

$$\mathcal{L} = \frac{\varepsilon_0}{2}\left[\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 - c^2 \left(\nabla \times \mathbf{A}\right)^2\right] = \frac{\varepsilon_0}{2}\left[\mathbf{E}^2 - c^2\mathbf{B}^2\right], \tag{9.3}$$

where

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}. \tag{9.4}$$

The Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta \mathbf{A}_i} - \frac{\partial}{\partial t}\frac{\delta \mathcal{L}}{\delta\left(\frac{\partial \mathbf{A}_i}{\partial t}\right)} - \sum_j \frac{\partial}{\partial \mathbf{x}_j}\frac{\delta \mathcal{L}}{\delta\left(\frac{\partial \mathbf{A}_i}{\partial x_j}\right)} = 0 \tag{9.5}$$

lead to Maxwell's equations:

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \tag{9.6}$$

$$\nabla \cdot \mathbf{E} = 0. \tag{9.7}$$

From Equations (9.6),(9.7) one can get the wave equation:

$$\left[\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right]\mathbf{E} = 0. \tag{9.8}$$

Now calculate the canonical momentum $\mathbf{P}$ from Equation (9.1):

$$\mathbf{P}_i = \frac{\delta \mathcal{L}}{\delta\left(\frac{\partial \mathbf{A}_i}{\partial t}\right)} \qquad \Rightarrow \qquad \mathbf{P} = -\varepsilon_0\mathbf{E}. \tag{9.9}$$

According to

$$\mathcal{H} = P \cdot \frac{\partial \mathbf{A}}{\partial t} - \mathcal{L} \tag{9.10}$$

we can write the Hamiltonian density:

$$\mathcal{H} = \frac{1}{2\varepsilon_0}\mathbf{P}^2 + \frac{\varepsilon_0 c^2}{2}\left(\nabla \times \mathbf{A}\right)^2 = \frac{\varepsilon_0}{2}\left[\mathbf{E}^2 + c^2\mathbf{B}^2\right]. \tag{9.11}$$

As we already discussed, the naive approach for quantization, relation of the type (9.2) for $\left[\hat{\mathbf{A}}_l\left(\mathbf{r}, t\right), \hat{\mathbf{P}}_j\left(\mathbf{r}', t\right)\right]$, is not correct because the fields are not unconstrained variables. One could try to express the Lagrangian density using only two independent components of $\mathbf{A}$ instead of three, and define conjugate variables, but lot of the intuition that we obtain when keeping all components of the fields would be lost. Therefore, the quantization of the field is done by keeping all components of $\mathbf{A}$, with the consequence that the commutation relations will actually look a bit different from the ones we are used to in case of unconstrained variables such as position and momentum. Let us start by looking at the classical solution for the electromagnetic field and postulating that the amplitudes therein should be quantized.

In **Exercise Set 2, Task 5.** you considered the plane wave solution of the electromagnetic field (this is a solution of the wave equation (9.8))

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},\lambda} \mathbf{e}_{\mathbf{k}\lambda} \left[A_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + A^*_{\mathbf{k}\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} \right] \tag{9.12}$$

Now this is the plane wave solution of the electromagnetic field: however, eigenmodes of classical electromagnetic field can be in general more complicated depending on the geometry, think for instance about light inside an optical fiber or in a cavity, etc. We can thus introduce a more general quantity $\mathbf{u}_{\mathbf{k}\lambda}$ that contains the polarisation vector and the spatial dependence of the possibly complicated shape eigenfunction (in general $\mathbf{k}$ refers to the eigenmode and does not need to be the momentum as it is in the plane wave and similar cases). For plane waves, one has

$$\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) = \frac{1}{\sqrt{V}} \mathbf{e}_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{9.13}$$

where $\mathbf{e}_{\mathbf{k}\lambda}$ is the polarisation vector. Now let us write the solution of the wave equation (9.8) using the more general eigenfunctions:

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \left[A_{\mathbf{k}\lambda} \mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) e^{-i\omega_{\mathbf{k}}t} + A^*_{\mathbf{k}\lambda} \mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r}) e^{i\omega_{\mathbf{k}}t} \right]. \tag{9.14}$$

We now do the quantization by stating/postulating that the amplitudes $A_{\mathbf{k}\lambda}$ are actually operators, $\hat{A}_{\mathbf{k}\lambda}$, equivalent to bosonic annihilation operators up to a normalization constant: $\hat{A}_{\mathbf{k}\lambda} = \mathcal{N}\hat{a}_{\mathbf{k}\lambda}$ where

$$\begin{aligned} \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] &= \delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'} \\ \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}\right] &= \left[\hat{a}^\dagger_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] = 0. \end{aligned} \tag{9.15}$$

The normalization $\mathcal{N}$ is chosen to lead to one photon per unit volume, cf. Lecture 2. This statement/postulate is backed by experimental observations such as the strict validity of Planck's law which assumes quanta of energy, and the intrinsic angular momentum of electromagnetic radiation being $S = 1$ (the spin-statistics theorem then tells that the quanta of the field are bosons).

Then, we can write the solutions for the vector potential and the electric field as operators:

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}\varepsilon_0}} \left[\hat{a}_{\mathbf{k}\lambda} \mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) e^{-i\omega_{\mathbf{k}}t} + \hat{a}^\dagger_{\mathbf{k}\lambda} \mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r}) e^{i\omega_{\mathbf{k}}t} \right] \tag{9.16}$$

$$\hat{\mathbf{E}}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\hat{\mathbf{P}} = i \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\varepsilon_0}} \left[\hat{a}_{\mathbf{k}\lambda} \mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) e^{-i\omega_{\mathbf{k}}t} - \hat{a}^\dagger_{\mathbf{k}\lambda} \mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r}) e^{i\omega_{\mathbf{k}}t} \right]. \tag{9.17}$$

Remember that $\mathbf{u}_{\mathbf{k}\lambda}$ are orthonormal eigensolutions of the Maxwell's equations in the Coulomb gauge

$$\int_V \mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r})\,\mathbf{u}_{\mathbf{k}'\lambda'}(\mathbf{r})\,d\mathbf{r} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'} \tag{9.18}$$

$$\left(\nabla^2 + \frac{\omega^2_{\mathbf{k}}}{c^2}\right)\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) = 0 \tag{9.19}$$

$$\nabla\cdot\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) = 0 \qquad \left(\nabla\cdot\hat{\mathbf{A}} = 0\right) \tag{9.20}$$

so that for instance for the plane wave

$$\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) = \frac{1}{\sqrt{L^3}}\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{9.21}$$

one has

$$\mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{k} = 0 \qquad \left(\nabla\cdot\hat{\mathbf{A}} = 0\right), \tag{9.22}$$

which means that the polarisation directions are orthogonal to the propagation direction.

Then the total Hamiltonian (9.11) can be expressed using (9.15) and (9.16)–(9.20) in a very simple form (the calculation is similar to the one done in **Exercise Set 2, Task 5.**):

$$\hat{H} = \int d\mathbf{r}\mathcal{H} = \sum_{\mathbf{k},\lambda}\hbar\omega_{\mathbf{k}}\left(\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} + \frac{1}{2}\right). \tag{9.23}$$

Finally, one can then derive the commutation relation for the fields $\hat{\mathbf{A}}$ and $\hat{\mathbf{P}}$. It is an exercise for you (**Exercise Set 5**) to show that one obtains

$$\left[\hat{\mathbf{A}}_l(\mathbf{r},t), \hat{\mathbf{P}}_j(\mathbf{r}',t)\right] = -\varepsilon_0\left[\hat{\mathbf{A}}_l(\mathbf{r},t), \hat{\mathbf{E}}_j(\mathbf{r}',t)\right] = i\hbar\left(\delta_{lj} - \frac{\partial^l\partial^j}{\nabla^2}\right)\delta(\mathbf{r}-\mathbf{r}'). \tag{9.24}$$

The commutator therefore becomes proportional to the transverse delta function

$$\begin{aligned}\delta^{\perp}_{lj}(\mathbf{r}-\mathbf{r}') &= \left(\delta_{lj} - \frac{\partial^l\partial^j}{\nabla^2}\right)\delta(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\,\delta_{lj} + \frac{1}{4\pi}\partial^l\partial^j\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\\ &= \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left(\delta_{lj} - \frac{k_l k_j}{k^2}\right).\end{aligned} \tag{9.25}$$

Here V is the system volume.

For a student that wants to have deeper understanding and a more elegant treatment of the quantization of the field (not part of the course), the chapters 7.6, 8.2 and 8.3 of S. Weinberg, Quantum Theory of Fields, vol. 1: Foundations, Cambridge University Press 1995 are recommended. Also the chapters II and III of C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, Photons and Atoms: Introduction to Quantum Electrodynamics, Wiley 1997 can be useful.

Since now the field is quantized, there will be uncertainty relations (the Heisenberg uncertainly relation) between quantities related to the field. From elementary quantum mechanics, we are used to the idea of having an uncertainty relation between position and momentum. But of course one can always define new operators which are linear combinations of position and momentum operators, by doing a unitary transformation, and they will then have corresponding uncertainty relations. Similarly, for fields, one can define various uncertainty relations. A typical one is the uncertainty relation of amplitude A (related to photon number) and phase φ

$$\Delta A \Delta \varphi \geq \frac{\hbar}{2}. \tag{9.26}$$

This means that if the photon number is perfectly known, then phase is undefined, and vice versa. More generally, one can define the so-called **quadratures** $\hat{q}$ and $\hat{p}$ which can be for instance of the form (a more general form is given later)

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}} \left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{p} = -i\sqrt{\frac{\hbar\omega}{2}} \left(\hat{a} - \hat{a}^\dagger\right). \tag{9.27}$$

The quadratures obey the uncertainty relation. Formally, they resemble the position and momentum operators in case of a harmonic oscillator. In case of electromagnetic field, they correspond to its real and imaginary part (we forget here the spatial dependence, the constants in front, and summations over momenta):

$$\begin{aligned} \hat{\mathbf{E}} &\propto \hat{a}\exp\left(-i\omega t\right) - \hat{a}^\dagger \exp\left(i\omega t\right) \\ &= (\hat{a} + \hat{a}^\dagger)\sin\left(\omega t\right) + i(\hat{a} - \hat{a}^\dagger)\cos\left(\omega t\right) \\ &= \hat{q}\sin\left(\omega t\right) + \hat{p}\cos\left(\omega t\right). \end{aligned}$$

Here we have used a normalization where $\hbar/(2\omega)$ in Eq. (9.35) is one. The appendix of this lecture shows how the quadratures could be measured.
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9.2 Quantum states of the electromagnetic field

The many-body system of photons can be in various different quantum states.

9.2.1 Fock state $|n_\mathbf{k}\rangle$

One useful basic state (which also can be realized experimentally at least for low numbers of photons) is the Fock state (also called number state):

$$\hat{n}_\mathbf{k} = \hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{k} \tag{9.28}$$

$$\hat{n}_\mathbf{k}\left|n_\mathbf{k}\right\rangle = n_\mathbf{k}\left|n_\mathbf{k}\right\rangle. \tag{9.29}$$

The Fock state can be generated from the vacuum state in the following way:

$$\hat{a}_\mathbf{k}\left|0\right\rangle = 0 \tag{9.30}$$

$$\hat{a}_\mathbf{k}\left|n_\mathbf{k}\right\rangle = \sqrt{n_\mathbf{k}}\left|n_\mathbf{k} - 1\right\rangle \tag{9.31}$$

$$\hat{a}_\mathbf{k}^\dagger\left|n_\mathbf{k}\right\rangle = \sqrt{n_\mathbf{k} + 1}\left|n_\mathbf{k} + 1\right\rangle \tag{9.32}$$

So:

$$\left|n_\mathbf{k}\right\rangle = \frac{\left(\hat{a}_\mathbf{k}^\dagger\right)^{n_\mathbf{k}}}{\sqrt{n_\mathbf{k}!}}\left|0\right\rangle. \tag{9.33}$$

Fock states can be created experimentally, for instance, by having a certain (small) number of photons in an optical cavity, or a single photon travelling in an optical fiber. Note that since the photon number is totally fixed, the phase of the field in a Fock state is completely random, following from the Heisenberg uncertainty relation. Fock states are very strongly "quantum", with no close classical counterpart.

9.2.2 Coherent state $|\alpha\rangle$

Coherent states are the closest quantum states to the classical description of the field. For instance the laser light is typically in a coherent state. For a coherent state, the uncertainty (as given by the Heisenberg uncertainty relation) is equal in the phase φ and the amplitude A of the field.

$$\Delta A \Delta\varphi = \frac{\hbar}{2}, \qquad \Delta A = \Delta\varphi \tag{9.34}$$

Let us remind the definition of the quadratures:

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right). \tag{9.35}$$

To define the coherent state, one demands that the harmonic oscillator classical energy is the same as the expectation value of the Hamiltonian:

$$H_{cl} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = \text{ (for mean values) } = \frac{1}{2}\left[\langle\alpha|\,\hat{p}\,|\alpha\rangle^2 + \omega^2\langle\alpha|\,\hat{q}\,|\alpha\rangle^2\right]$$

which is, using Equation (9.35),

$$H_{cl} = \hbar\omega\,\langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle\,\langle\alpha|\,\hat{a}\,|\alpha\rangle \tag{9.36}$$

we demand this to be the same as the expectation value of the quantum Hamiltonian, which is (we neglect $\frac{1}{2}\hbar\omega$ by shifting the energy)

$$\langle\alpha|\,\hat{H}\,|\alpha\rangle = \hbar\omega\,\langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle\,. \tag{9.37}$$

Our demand thus sets

$$\hbar\omega\,\langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle\,\langle\alpha|\,\hat{a}\,|\alpha\rangle = \hbar\omega\,\langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle\,. \tag{9.38}$$

Equation (9.38) $\Rightarrow$

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{9.39}$$

$$\langle\alpha|\,\hat{a}^\dagger = \langle\alpha|\,\alpha^*. \tag{9.40}$$

Here $|\alpha|^2$ is the mean photon number

$$|\alpha|^2 = \frac{\langle\alpha|\,\hat{H}\,|\alpha\rangle}{\hbar\omega}. \tag{9.41}$$

In order to express the coherent state in terms of Fock states, we consider

$$\alpha \langle n | \alpha \rangle = \langle n | (\hat{a} |\alpha\rangle) = (\langle n | \hat{a}) |\alpha\rangle = \sqrt{n+1} \langle n+1 | \alpha \rangle \tag{9.42}$$

$\Rightarrow$

$$\langle n+1 | \alpha \rangle = \frac{\alpha}{\sqrt{n+1}} \langle n | \alpha \rangle \tag{9.43}$$

By iteration one obtains

$$\langle n | \alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} \langle 0 | \alpha \rangle . \tag{9.44}$$

Finally, using for normalization the requirement

$$|\langle \alpha | \alpha \rangle|^2 = 1, \tag{9.45}$$

one gets the final result

$$|\alpha\rangle = \langle 0 | \alpha \rangle \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \tag{9.46}$$

Note: $P(n) = |\langle n | \alpha \rangle|^2$ is the Poisson distribution.

9.2.3 The Displacement Operator

The displacement operator creates a coherent state from the vacuum. It is defined as (c.f. **Exercise Set 5**)

$$\mathcal{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \tag{9.47}$$

$$\mathcal{D}(\alpha) |0\rangle = |\alpha\rangle . \tag{9.48}$$

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9.2.4 Squeezed state

Squeezed state resembles the coherent state, but now the uncertainties in two quadratures (like phase and amplitude) can be different: the fluctuations in one quadrature are suppressed below the standard quantum noise limit, while in the other one they are increased.

Now denote the quadratures by X_1 and X_2

$$X_1 = \frac{1}{2}\left(a e^{i\phi} + a^\dagger e^{-i\phi}\right), \qquad X_2 = \frac{1}{2i}\left(a e^{i\phi} - a^\dagger e^{-i\phi}\right) \tag{9.49}$$

$$[X_1, X_2] = \frac{i}{2} \tag{9.50}$$

$$\hat{\mathbf{E}}(\mathbf{r}, t) \propto X_1 \sin(\omega t - \mathbf{k} \cdot \mathbf{r}) - X_2 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \text{; also like } \hat{q}, \hat{p} \tag{9.51}$$

Both coherent and squeezed states are **minimum uncertainty states**:

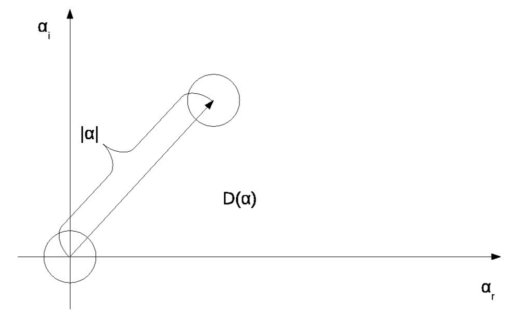

Figure 8: The displacement operator $D(\alpha)$ makes the field to have a non-zero amplitude $|\alpha|$.

$$\Delta X_1 \Delta X_2 = \frac{1}{4} \tag{9.52}$$

but
$\Delta X_1 = \Delta X_2$ for coherent states
$\Delta X_1 \neq \Delta X_2$ for squeezed states.

9.2.5 The Squeeze Operator

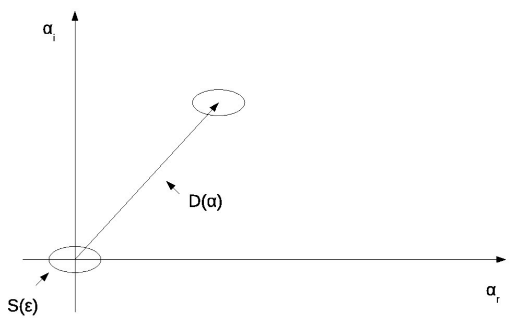

Figure 9: The squeeze operator $S(\epsilon)$ deforms the noise in the quadratures, and the displacement operator $D(\alpha)$ makes the field to have a non-zero amplitude $|\alpha|$.

Analogously to the displacement operator, the squeeze operator modifies the fluctuations (noise, uncertainty) in the two quadratures of the vacuum (or any coherent) state. The squeeze operator is defined as

$$S(\varepsilon) = \exp\left(\frac{1}{2}\varepsilon a^{\dagger 2} - \frac{1}{2}\varepsilon^* a^2\right) \tag{9.53}$$

$$\varepsilon = re^{-2i\phi} \tag{9.54}$$

A squeezed state can then be expressed as

$$\mathcal{D}(\alpha)\, S(\varepsilon)\, |0\rangle = |\alpha, \epsilon\rangle \text{ for } \phi = 0. \tag{9.55}$$

Note: The Fock state and the coherent state (sometimes even the squeezed state) are concepts applied widely also in other contexts than light-matter interactions. Coherent, squeezed and Fock states have been experimentally observed in modern quantum optics.

10 Example of a coherent state: a condensed weakly interacting Bose gas

As another example of second quantization we now study a weakly interacting gas of bosonic particles. Later in the course, in Lecture 8, you will learn more about ultracold gases, where Bose Einstein condensates of weakly interacting atoms such as ^{85}Rb or ^{7}Li have been made. Let us use the Hamiltonian in the momentum space (c.f. previous lecture), and assume that the two-particle interaction strength does not depend on momentum, that is $V(q) = g$, where g is some constant. Now

$$H = H_0 + H_{\text{int}} = \sum_k \frac{\hbar^2 k^2}{2m} a_k^\dagger a_k + \frac{g}{2V} \sum_{kpq} a_k^\dagger a_{-k+q}^\dagger a_{-p+q} a_p. \tag{10.1}$$

At low enough temperatures, in a thermodynamic equilibrium, majority of atoms will be *condensed* in the ground state, i.e. in the zero-momentum single-particle state $|p = 0\rangle$ (you will learn more about Bose Einstein condensation in Lecture 8). The atoms in this state can be described well by the **coherent state** $|\alpha\rangle$. The basic property of a coherent state is the relation $a|\alpha\rangle = \alpha|\alpha\rangle$. The coherent state motivates a huge simplification suggested by Bogoliubov: choosing $\alpha = \sqrt{N_0}$, where N_0 is the (expectation value of) number of atoms in the condensate, we obtain $a_0|\alpha\rangle = \alpha|\alpha\rangle = \sqrt{N_0}|\alpha\rangle$. Thus if we simply replace the operator a_0 by a complex number $\sqrt{N_0}$, this relation remains unchanged. And even though $|\alpha\rangle$ is not an eigenstate of $a_0^\dagger$, we replace also $a_0^\dagger \to \sqrt{N_0}$. Of course, this is an approximation and is based on N_0 being large (which justifies that the opertors corresponding to N_0, i.e., a_0, $a_0^\dagger$, are replaced by complex numbers). The total number of atoms is $N = \langle \hat{N} \rangle = N_0 + \sum_{k \neq 0} \langle a_k^\dagger a_k \rangle$. These approximations allow deriving various analytical results, for instance the so-called Bogoliubov spectrum of excitations which predicts that the weakly interacting BEC will have a collective (sound) mode at low energies. There are also many other examples in physics of systems where the coherent state is a powerful concept in describing the main phenomena.

The Bogoliubov theory of a BEC is an example of a **mean-field theory**, in which some quantum operators (those corresponding to the occupation of the lowest energy state in the case of BEC) are replaced by complex numbers. Mean-field theories often allow an efficient description of many-body problems. However, the mean-field theory does come with a cost and often they result in Hamiltonians that do not conserve the number of particles. In some cases, mean-field descriptions miss some essential features of the physics of the system, and a more advanced description is needed.

Appendix: A page from Mandel and Wolf, Optical Coherence and Quantum Optics, Cambridge University Press 1995, showing an example how the quadratures can be measured by mixing the signal with a classical field whose amplitude and phase are known and controlled precisely.

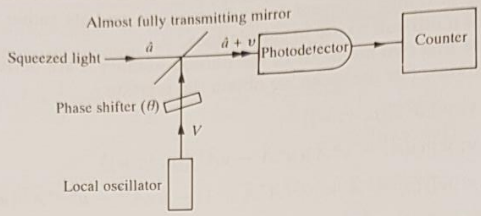

Fig. 21.4 Detection of a squeezed state by interference or homodyning with the coherent light beam of a local oscillator.

then falls on a photodetector that counts the number n of photons detected in some time interval short compared with the coherence time. The moments of n can be determined by repeating the measurement many times. The object of the experiment is to vary the local oscillator phase θ so as to identify the squeezed field quadrature, which manifests itself in reduced fluctuations of the combined light intensity.

This interference or homodyne problem has been treated several times (Yuen and Shapiro, 1980; Shapiro, Yuen and Machado Mata, 1979; Mandel, 1982a). Here we shall largely follow the last treatment. The field amplitude at the detector is proportional to $\hat{a}+v$. Hence the average number of counts $\langle\hat{n}\rangle$ registered in some short time interval is given by the expression

$$\langle\hat{n}\rangle = \eta\langle(\hat{a}^\dagger + v^*)(\hat{a}+v)\rangle, \qquad (21.6\text{–}1)$$

and for the second factorial moment of the number $\hat{n}$ we have, since $\hat{n}^{(2)} = :\hat{n}^2:$,

$$\langle\hat{n}(\hat{n}-1)\rangle = \eta^2\langle(\hat{a}^\dagger + v^*)^2(\hat{a}+v)^2\rangle. \qquad (21.6\text{–}2)$$

The parameter η is a measure of the detector efficiency and of the counting time. We first use Eqs. (21.1–7) to express $\hat{a}$, $\hat{a}^\dagger$ in terms of the quadrature amplitudes $\hat{Q}$, $\hat{P}$,

$$\left.\begin{aligned}\hat{a} &= \tfrac{1}{2}(\hat{Q}+\mathrm{i}\hat{P})\,\mathrm{e}^{\mathrm{i}\beta}\\ \hat{a}^\dagger &= \tfrac{1}{2}(\hat{Q}-\mathrm{i}\hat{P})\,\mathrm{e}^{-\mathrm{i}\beta}.\end{aligned}\right\} \qquad (21.6\text{–}3)$$

It follows that

$$\begin{aligned}\langle\hat{n}\rangle &= \eta\langle[\tfrac{1}{2}(\hat{Q}-\mathrm{i}\hat{P})\,\mathrm{e}^{-\mathrm{i}\beta}+v^*][\tfrac{1}{2}(\hat{Q}+\mathrm{i}\hat{P})\,\mathrm{e}^{\mathrm{i}\beta}+v]\rangle\\ &= \eta\{\tfrac{1}{4}[\langle\hat{Q}^2\rangle + \langle\hat{P}^2\rangle + \mathrm{i}\langle\hat{Q}\hat{P}-\hat{P}\hat{Q}\rangle] + [\tfrac{1}{2}(\langle\hat{Q}\rangle - \mathrm{i}\langle\hat{P}\rangle)v\,\mathrm{e}^{-\mathrm{i}\beta} + \text{c.c.}] + |v|^2\},\end{aligned}$$

and with the help of the commutation relation $[\hat{Q},\hat{P}]=2\mathrm{i}$, this becomes, if $v=|v|\,\mathrm{e}^{\mathrm{i}\theta}$,

$$\langle\hat{n}\rangle = \eta\{\tfrac{1}{4}(\langle\hat{Q}^2\rangle + \langle\hat{P}^2\rangle - 2) + |v|[\langle\hat{Q}\rangle\cos(\theta-\beta) + \langle\hat{P}\rangle\sin(\theta-\beta)] + |v|^2\}. \qquad (21.6\text{–}4)$$

Similarly we may evaluate the second factorial moment $\langle\hat{n}(\hat{n}-1)\rangle$. Finally we combine the results to determine the departure from Poisson statistics and we obtain, after some rearrangement of terms,