

## Lecture 6

### Learning goals

- To know how to describe the interaction of a two-level system with quantized field: to be able to derive the eigenstates of the system.
- To learn the following concepts: rotating wave approximation, Jaynes-Cummings model, Rabi splitting, Rabi oscillations, strong coupling.
- To understand how spontaneous emission is related to quantization of the field.

## 11 Quantized field interacting with a two-level system

*Literature:* P. Törmä and W.L. Barnes, Reports on Progress in Physics 78, 013901 (2015), <http://iopscience.iop.org/article/10.1088/0034-4885/78/1/013901> (to have access to the article, you might have to use a computer at Aalto domain; but you can try from a home computer too, it used to be a free article), note that the article has also a nice video abstract; P. Meystre and M. Sargent III, Elements of quantum optics (Springer), Chapters 13.1-13.2 (note that the detuning  $\delta$  there is defined in a non-standard way, namely the other way round than in this lecture and the above review article, which causes minor differences).

Two-level system: e.g. an electronic transition in an atom or in a molecule; a “qubit” in quantum information, etc. We denote the excited state by  $|e\rangle$  and the ground state by  $|g\rangle$ . One of the states does not need to be the actual ground state, just any two (excited) states can be chosen (then,  $|g\rangle$  marks the state with a lower energy and  $|e\rangle$  the higher energy). Note that, naturally, the **two-level approximation** has to be valid, i.e. the frequency of the field is reasonably close to the energy difference between  $|g\rangle$  and  $|e\rangle$ , but not close to the energy difference between any other pair of states.

The Hamiltonian is of the form (c.f. Lecture 2)

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad (11.1)$$

where  $H_0$  contains the energy of the two-level system:

$$\hat{H}_0 = E_e |e\rangle \langle e| + E_g |g\rangle \langle g| \quad (11.2)$$

and  $\hat{V}$  describes the interaction with the field  $\hat{\mathbf{E}}(\mathbf{R}, t)$  and is of the form

$$\hat{V} = -e \langle e | \hat{\mathbf{r}} | g \rangle \cdot \hat{\mathbf{E}}(\mathbf{R}, t) |e\rangle \langle g| - e \langle g | \hat{\mathbf{r}} | e \rangle \cdot \hat{\mathbf{E}}(\mathbf{R}, t) |g\rangle \langle e|. \quad (11.3)$$

We denote

$$\omega_{eg} = \frac{E_e - E_g}{\hbar}. \quad (11.4)$$

This interaction Hamiltonian can be derived from microscopic physics of an electron (in an atom) interacting with electromagnetic field, just like in Lecture 2. Of course, here the electric field is an operator, but that does not influence the derivation of Equation (11.3). You can do the derivation yourself starting from Equation (5.1) and remembering that  $\hat{\mathbf{E}}(\mathbf{R}, t) = -\partial \hat{\mathbf{A}}(\mathbf{R}, t) / \partial t = i\omega \hat{\mathbf{A}}(\mathbf{R}, t)$  for harmonic fields. Note also that here  $e$  means the elementary charge ( $e > 0$ ). The interaction  $\hat{V}$

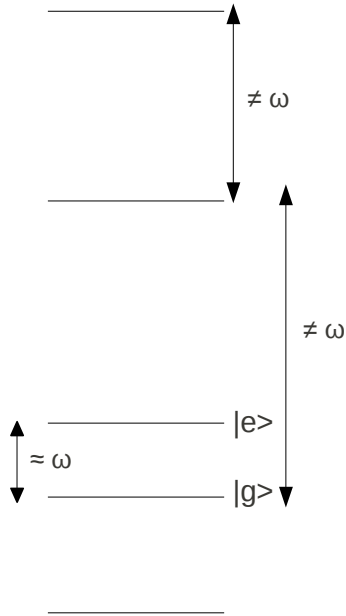


Figure 10: The two-level approximation: the energy  $\hbar\omega$  corresponding to the field of frequency  $\omega$  is close to only one of the possible transition energies.

is actually proportional to  $\omega_{eg}/\omega$ , where  $\omega$  is the frequency of the field, but since within the two-level approximation  $\omega_{eg}$  and  $\omega$  are quite close to each other, this ratio has been approximated to one and is not visible in Equation (11.3). (Remember the discussion in the end of Section 5.3: the microscopic Hamiltonian contained the term  $\hat{\mathbf{p}} \cdot \mathbf{A}$  and going from  $\hat{\mathbf{p}}$  to  $\hat{\mathbf{r}}$  produced  $\omega_{eg}$  which was then taken to be essentially the same as  $\omega$ .)

Note: now the electric field  $\hat{\mathbf{E}}(\mathbf{R}, t)$  is an operator, because the field is quantized! This is different from Lecture 2 where the field was just a complex vector. The electron position  $\hat{\mathbf{r}}$  is an operator, as also in Lecture 2, since the electron motion is treated quantum mechanically. Remember also from Lecture 2 that  $\mathbf{R}$  is the center-of-mass position of the atom, due to the **dipole approximation**.

**The dipole moment** is defined by

$$\vec{\mu}_{ge} = \langle e | -e\hat{\mathbf{r}} | g \rangle = -e \int d^3\mathbf{r} \mathbf{r} \phi_e^*(\mathbf{r}) \phi_g(\mathbf{r}) = \vec{\mu}_{eg}^* \quad (11.5)$$

Note that the dipole moments are vectors. In case of a centrosymmetric system, one has  $|\phi_g(\mathbf{r})|^2 = |\phi_g(-\mathbf{r})|^2$ ,  $|\phi_e(\mathbf{r})|^2 = |\phi_e(-\mathbf{r})|^2$  and thus

$$\vec{\mu}_{gg} = \vec{\mu}_{ee} = 0. \quad (11.6)$$

Now we can write the interaction term in the Hamiltonian as

$$\hat{V} = \overrightarrow{\mu}_{ge} \cdot \hat{\mathbf{E}} |e\rangle \langle g| + \overrightarrow{\mu}_{eg} \cdot \hat{\mathbf{E}} |g\rangle \langle e|, \quad (11.7)$$

or, using the auxiliary operators  $\hat{\sigma}_+ = |e\rangle \langle g|$ ,  $\hat{\sigma}_- = |g\rangle \langle e|$ :

$$\hat{V} = \overrightarrow{\mu}_{ge} \cdot \hat{\mathbf{E}} \hat{\sigma}_+ + \overrightarrow{\mu}_{eg} \cdot \hat{\mathbf{E}} \hat{\sigma}_-. \quad (11.8)$$

In the interaction picture we have (c.f. Lecture 1; start from  $\hat{V}_I = e^{i\frac{H_0 t}{\hbar}} \hat{V} e^{-i\frac{H_0 t}{\hbar}}$  and use Equation (11.2) and the definition (11.4), see **Exercise Set 6**)

$$\hat{V}_I = \overrightarrow{\mu}_{ge} \cdot \hat{\mathbf{E}} |e\rangle \langle g| e^{i\omega_{eg} t} + h.c.. \quad (11.9)$$

Now include the quantized field

$$\hat{\mathbf{E}}(\mathbf{R}, t) = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0}} [\hat{\mathbf{a}}\mathbf{u}(\mathbf{R}) e^{-i\omega t} - \hat{\mathbf{a}}^\dagger \mathbf{u}^*(\mathbf{R}) e^{i\omega t}] \quad (11.10)$$

and choose

$$\overrightarrow{\mu}_{ge} \parallel \mathbf{u}(\mathbf{R}), \quad \mathbf{u}(\mathbf{R}) = \mathbf{u}^*(\mathbf{R}), \quad g \equiv i\sqrt{\frac{\omega}{2\hbar\varepsilon_0}} \overrightarrow{\mu}_{ge} \cdot \mathbf{u}(\mathbf{R}). \quad (11.11)$$

If initially we did not approximate  $\omega_{eg}/\omega \simeq 1$ , then the coupling constant  $g$  would have the form

$$g \equiv i\frac{\omega_{eg}}{\sqrt{2\hbar\varepsilon_0\omega}} \overrightarrow{\mu}_{ge} \cdot \mathbf{u}(\mathbf{R}). \quad (11.12)$$

This is useful to remember when comparing definitions of  $g$  in different textbooks. The Equation (11.9) becomes

$$\hat{V}_I = \hbar (\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) (g \hat{\sigma}_+ e^{i\omega_{eg} t} + g^* \hat{\sigma}_- e^{-i\omega_{eg} t}). \quad (11.13)$$

This has four terms oscillating at two different frequencies:

$$\begin{aligned} \hat{V}_I &= \hbar \left( g \hat{a} \hat{\sigma}_+ e^{-i(\omega - \omega_{eg})t} + g^* \hat{a}^\dagger \hat{\sigma}_- e^{i(\omega - \omega_{eg})t} \right) \\ &+ \hbar \left( g^* \hat{a} \hat{\sigma}_- e^{-i(\omega + \omega_{eg})t} + g \hat{a}^\dagger \hat{\sigma}_+ e^{i(\omega + \omega_{eg})t} \right). \end{aligned} \quad (11.14)$$

Since  $\omega$  is chosen to be close to  $\omega_{eg}$  (the two-level approximation), the terms oscillating at the frequency  $\omega - \omega_{eg}$  give the slow, relevant dynamics of the problem, whereas the terms oscillating at the high frequency  $\omega + \omega_{eg}$  (which is basically an optical frequency, i.e.  $10^{15}$  Hz) average out and these terms can be neglected. Another way of saying this is that the terms proportional to  $\omega - \omega_{eg}$  conserve energy: the term  $\hat{a} \hat{\sigma}_+$  means that a photon is destroyed (absorbed) and the atom is raised from the ground state to the excited, and the term  $\hat{a}^\dagger \hat{\sigma}_-$  means that a photon is created (emitted) and the atom is lowered from the excited to the ground state. The other two terms, corresponding to absorption with excited-ground transition, and emission with ground-excited transition, are the non-resonant, i.e. not energy conserving terms, and are often negligible. As discussed already in Lecture 1 (3.3. Harmonic perturbation), these terms (sometimes also called counter-rotating terms) are neglected. This is called the **Rotating Wave Approximation (RWA)**. Within RWA, the Hamiltonian becomes

$$\hat{V}_I = \hbar \left( g \hat{a} \hat{\sigma}_+ e^{-i(\omega - \omega_{eg})t} + g^* \hat{a}^\dagger \hat{\sigma}_- e^{i(\omega - \omega_{eg})t} \right). \quad (11.15)$$

Going from the interaction picture back to the total Hamiltonian (i.e. relating the terms  $e^{\pm i(\omega - \omega_{eg})t}$  to a Hamiltonian  $\hat{H}_0$ ), one obtains (see **Exercise Set 6**) the so-called Jaynes-Cummings Hamiltonian of the **the Jaynes-Cummings model**:

$$\hat{H} = \frac{1}{2} \hbar \omega_{eg} \hat{\sigma}_z + \hbar \omega \hat{a}^\dagger \hat{a} + \hbar (g \hat{a} \hat{\sigma}_+ + h.c.). \quad (11.16)$$

That is,

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad (11.17)$$

where

$$\hat{H}_0 = \frac{1}{2} \hbar \omega_{eg} \hat{\sigma}_z + \hbar \omega \hat{a}^\dagger \hat{a} \quad (11.18)$$

Here  $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$  and (note the shift in  $\omega_{eg}$ ; constants have been neglected from the Hamiltonian),

$|e\rangle$  \_\_\_\_\_  
 .....  
 $|g\rangle$  \_\_\_\_\_

$$\hat{V} = \hbar (g \hat{a} \hat{\sigma}_+ + h.c.). \quad (11.19)$$

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Let us now use for the quantized field the Fock states as a basis. One notes that  $\hat{V}$  only couples the states  $|e\rangle |n\rangle$  and  $|g\rangle |n+1\rangle$ , i.e. one photon is emitted/absorbed when the atom makes a transition between the ground and the excited states. Therefore one can write the Hamiltonian as (remember that the state of the light field may have a distribution of photon numbers)

$$\hat{H} = \sum_n \hat{H}_n. \quad (11.20)$$

In the basis  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |e\rangle |n\rangle$ ,  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |g\rangle |n+1\rangle$ , the Hamiltonian  $\hat{H}_n$  is

$$\hat{H}_n = \hbar \left( n + \frac{1}{2} \right) \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} -\delta & 2g\sqrt{n+1} \\ 2g^*\sqrt{n+1} & \delta \end{pmatrix} \quad (11.21)$$

$$\delta = \omega - \omega_{eg}. \quad (11.22)$$

Here  $\delta = \omega - \omega_{eg}$  is called the detuning.

Note that if the field is not quantized, i.e. in the semi-classical description, one can follow the same procedure as we do now, but just replace  $2g\sqrt{n+1}$  by the semi-classical Rabi frequency  $\Omega$  which is proportional  $\mathbf{d} \cdot \mathbf{E}$ , i.e. to the dipole moment and the field amplitude. We will come back to this comparison later.

Now diagonalizing Equation (11.21) one obtains the following eigenvalues (NOTE: from now on, we re-define the notation so that we replace  $|g|$  by  $g$ ):

$$E_{1n} = \hbar \left( n + \frac{1}{2} \right) \omega - \frac{1}{2} \hbar \sqrt{\delta^2 + 4g^2 (n+1)} \quad (11.23)$$

$$E_{2n} = \hbar \left( n + \frac{1}{2} \right) \omega + \frac{1}{2} \hbar \sqrt{\delta^2 + 4g^2 (n+1)}, \quad (11.24)$$

where

$$\mathcal{R}_n = \sqrt{\delta^2 + 4g^2 (n+1)} \quad (11.25)$$

is the generalised Rabi frequency. This form of eigenstates is very important and general! It comes out always when two levels (states) of a quantum system are coupled, and they have some energy difference  $\delta$ . What comes from the quantization of the field, is the specific form of the term  $\sqrt{\delta^2 + 4g^2 (n+1)}$ , i.e. how it depends on the photon number  $n$ . We will come back to this when discussing spontaneous emission. For writing down the eigenstates, we denote

$$\cos \theta_n = \frac{\mathcal{R}_n - \delta}{\sqrt{(\mathcal{R}_n - \delta)^2 + 4g^2 (n+1)}} \quad (11.26)$$

$$\sin \theta_n = \frac{2g\sqrt{n+1}}{\sqrt{(\mathcal{R}_n - \delta)^2 + 4g^2 (n+1)}}. \quad (11.27)$$

With this definition, the eigenstates (the so-called **dressed states**) are

$$|1n\rangle = -\sin \theta_n |e\rangle |n\rangle + \cos \theta_n |g\rangle |n+1\rangle \quad (11.28)$$

$$|2n\rangle = \cos \theta_n |e\rangle |n\rangle + \sin \theta_n |g\rangle |n+1\rangle. \quad (11.29)$$

To understand the result intuitively, let us consider some limiting cases.

Case 1:

On resonance, i.e. when the field and the transition energy are the same ( $\delta = 0$ ), we have

$$\delta = 0 \quad (11.30)$$

$$|1n\rangle = \frac{1}{\sqrt{2}} [-|e\rangle |n\rangle + |g\rangle |n+1\rangle] \quad (11.31)$$

$$|2n\rangle = \frac{1}{\sqrt{2}} [|e\rangle |n\rangle + |g\rangle |n+1\rangle] \quad (11.32)$$

$$E_{1n} = \hbar \left( n + \frac{1}{2} \right) \omega + \hbar g \sqrt{n+1} \quad (11.33)$$

$$E_{2n} = \hbar \left( n + \frac{1}{2} \right) \omega - \hbar g \sqrt{n+1}. \quad (11.34)$$

This means that the eigenstates of the system are actually an equal superposition of the ground state + photon and the excited + no photon!

Case 2:

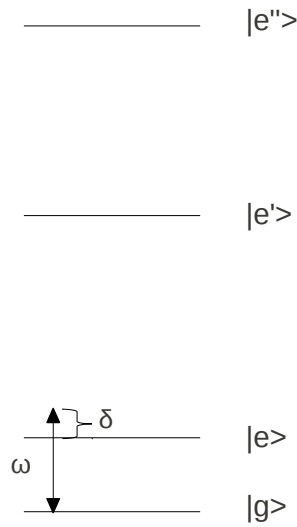


Figure 11: The two-level approximation and the far-detuned case: Note that even when the field frequency  $\omega$  can be far-detuned from the transition in question, i.e. the detuning  $\delta = \omega - \omega_{eg}$  is so large that one does not see resonance behaviour, the field frequency can still be so far from the other electronic transitions that the two-level approximation stays valid.

The field frequency deviates considerably from the transition frequency (however, it still has to be quite close to preserve the two-level approximation and the RWA); this is called the regime where the field is **far detuned**. Specifically, one can approximate by taking the limit  $|\delta| \gg g(n+1)$ . Then, if the field frequency is the bigger one (the so-called **blue-detuned** case)

$$\omega_{eg} < \omega \quad (11.35)$$

$$|1n\rangle \simeq -|e\rangle |n\rangle \quad (11.36)$$

$$|2n\rangle \simeq |g\rangle |n+1\rangle \quad (11.37)$$

$$E_{1n} \simeq \hbar \left( n + \frac{1}{2} \right) \omega - \frac{\hbar \delta}{2} - \frac{\hbar g^2 (n+1)}{\delta} \quad (11.38)$$

$$E_{2n} \simeq \hbar \left( n + \frac{1}{2} \right) \omega + \frac{\hbar \delta}{2} + \frac{\hbar g^2 (n+1)}{\delta}. \quad (11.39)$$

This means that  $|e\rangle |n\rangle$  becomes the ground state. For  $\omega_{eg} > \omega$  (the so-called **red-detuned** case), similar calculation shows that the ground state becomes  $|g\rangle |n+1\rangle$ .

### 11.1 The concept of avoided crossing and strong coupling

Let us have a look how the energies just obtained behave as a function of a parameter, now the detuning  $\delta$ . Without the coupling (i.e.  $g = 0$ ), the energies would simply cross, since the ground and excited state energies depend linearly on the detuning (c.f. the Hamiltonian (11.21)). However, when the coupling is on ( $g \neq 0$ ), the real eigenstates are the ones obtained by the diagonalization. The energy states now form an **avoided crossing**. This simply means that if the energies of two states would cross (as a function of some parameter), in the presence of a coupling between the states their energies actually avoid crossing, because there will be new eigenstates in the system.

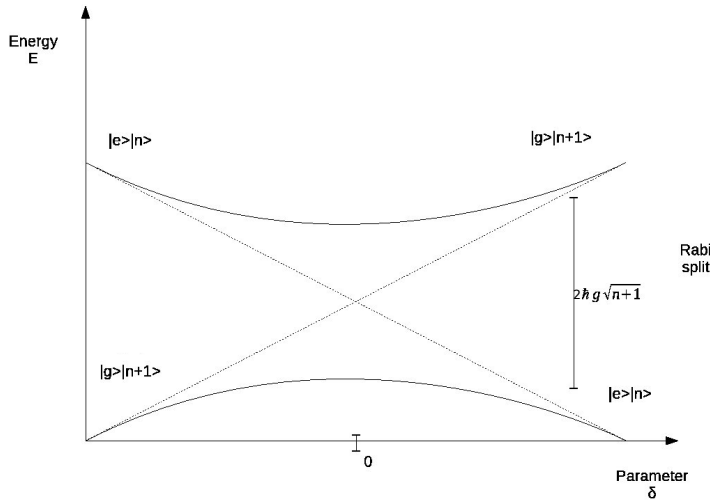


Figure 12: A quantized field interacting with a two-level system. Eigenenergies of the system as a function of the detuning  $\delta$ . Due to the interaction between the field and the two-level system, the energies form an avoided crossing: a Rabi split between the states occurs. If the split is bigger than the linewidths of the states, then the system is said to be in the strong coupling regime.

Note that the concept of avoided crossing does not require quantized field. It is enough to have two states that are coupled somehow. To see a remarkable avoided crossing, the coupling should be quite strong: the amount of the "split" or "splitting" between the states is directly proportional to the coupling between

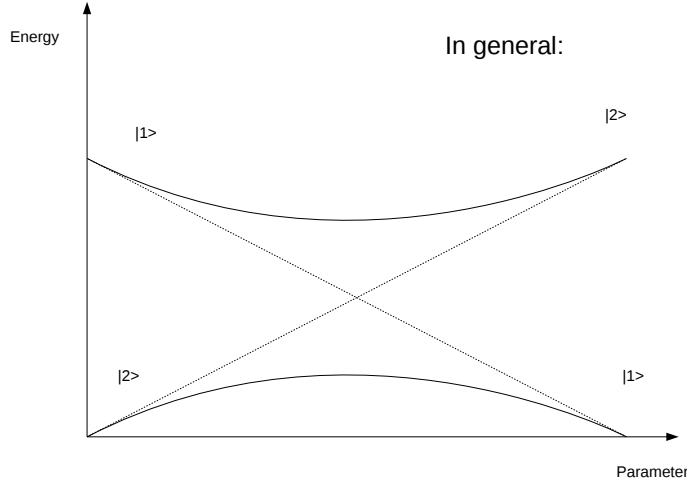


Figure 13: The general concept of an avoided crossing.

the states. The split, and thus the coupling, should be bigger than the linewidth of the states for the split to be visible. If the splitting is bigger than the linewidths, i.e. it can be experimentally resolved, the system is said to be in the **strong coupling** regime. Then the right physical picture describing the system are the new eigenstates, which means that the eigenstates (normal modes) are superpositions of the field excitations and the excitations of the atom. This physical picture and its mathematical description are totally different from the weak coupling regime where the Fermi Golden rule describes the system behaviour well.

By starting, e.g., from the state  $|1\rangle$  and sweeping the parameter  $\delta$  slowly (adiabatically), one can thus change the state from  $|1\rangle$  to  $|2\rangle$ . If one does the sweeping fast, there will be a transition over the crossing, the so-called Landau-Zener transition.

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## 11.2 Rabi oscillations

Since the Hamiltonian can be diagonalized exactly, we can calculate explicitly the time-development (dynamics) of the system. We will see the important phenomenon of **Rabi oscillations**. It is the time-domain counterpart of the Rabi split seen in the frequency (energy) domain. Let us apply a general ansatz using the eigenstates derived above:

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \left[ c_{1n}(0) e^{-i\frac{E_{1n}t}{\hbar}} |1n\rangle + c_{2n}(0) e^{-i\frac{E_{2n}t}{\hbar}} |2n\rangle \right] \quad (11.40)$$

Here  $c_{1n}(0)$  and  $c_{2n}(0)$  define the initial state. Furthermore, we define



$$c_{1n}(t) = c_{1n}(0) e^{-i \frac{E_{1n} t}{\hbar}} \quad (11.41)$$

$$c_{2n}(t) = c_{2n}(0) e^{-i \frac{E_{2n} t}{\hbar}} \quad (11.42)$$

The energies  $E_{2n}$  and  $E_{1n}$  are known. One can separate the terms  $\hbar(n + \frac{1}{2})\omega$  (they are the same in both eigenenergies) from the eigenenergies  $E_{1n}$  and  $E_{2n}$  and make a transformation where the whole wavefunction of the system has this term as a phase factor, in this way these terms can be transformed away; one can say that we are in a frame rotating with this frequency. Thus only the  $\mathcal{R}_n$  is left in the eigenenergies, and we have (writing the relations in a matrix form)

$$\begin{bmatrix} c_{2n}(t) \\ c_{1n}(t) \end{bmatrix} = \begin{bmatrix} e^{-i \frac{1}{2} \mathcal{R}_n t} & 0 \\ 0 & e^{i \frac{1}{2} \mathcal{R}_n t} \end{bmatrix} \begin{bmatrix} c_{2n}(0) \\ c_{1n}(0) \end{bmatrix}. \quad (11.43)$$

By applying the transformation between the dressed states and the original states (Equations (11.28) and (11.29)) this leads to

$$\begin{bmatrix} c_{en}(t) \\ c_{gn+1}(t) \end{bmatrix} = \begin{bmatrix} \cos \frac{1}{2} \mathcal{R}_n t + i \delta \mathcal{R}_n^{-1} \sin \frac{1}{2} \mathcal{R}_n t & -2ig\sqrt{n+1} \mathcal{R}_n^{-1} \sin \frac{1}{2} \mathcal{R}_n t \\ -2ig\sqrt{n+1} \mathcal{R}_n^{-1} \sin \frac{1}{2} \mathcal{R}_n t & \cos \frac{1}{2} \mathcal{R}_n t - i \delta \mathcal{R}_n^{-1} \sin \frac{1}{2} \mathcal{R}_n t \end{bmatrix} \begin{bmatrix} c_{en}(0) \\ c_{gn+1}(0) \end{bmatrix}. \quad (11.44)$$

In the case of the initial state  $|e\rangle|n\rangle$  and zero detuning, in other words, the case of:

$$\delta = 0, \quad c_{en}(0) = 1, \quad c_{gn+1}(0) = 0 \quad (11.45)$$

one obtains

$$|c_{en}(t)|^2 = \cos^2(g\sqrt{n+1}t) \quad (11.46)$$

$$|c_{gn+1}(t)|^2 = \sin^2(g\sqrt{n+1}t). \quad (11.47)$$

This time evolution is called Rabi oscillations.

*Note:* in the semiclassical analysis, one also obtains Rabi oscillations (Rabi flops). However, there  $g\sqrt{n+1}$  is replaced by the field amplitude (times the dipole moment). Therefore, in the limit of zero amplitude, i.e., no field, the Rabi oscillations, when starting from the excited state, disappear. This does not happen for the quantized field! Due to the 1 in the  $\sqrt{n+1}$ , there are Rabi oscillations also for  $n=0$ , i.e. zero photons in the initial field: an atom that is initially in the excited state, will emit a photon and come to the ground state. This is called **spontaneous emission**, and is totally an effect of the quantized field. In case of a classical field (with intensity zero), the two-level quantum atom would stay forever in the excited state if it is initially there.

In a fully classical theory, where the two-level atom is replaced by a classical Lorentzian oscillator and the field is also classical, one can derive damping of the oscillator due to so-called radiation reaction. This can then be thought of corresponding to spontaneous emission, and such a fully classical calculation gives roughly the correct decay time for the atom. However, we know from many experiments that atoms are rather two-level systems than Lorentzian oscillators (for instance saturation, i.e. that you cannot excite the atom more once it is in the

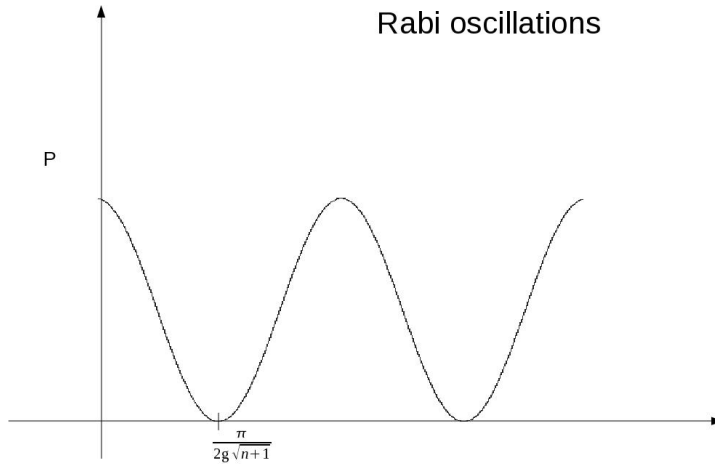


Figure 14: Rabi oscillations

excited state, is a clear indication of this). When taking the atom as a two-level system, only quantized field can predict the decay of the excited state even in presence of no field. This spontaneous emission is said to be due to quantum fluctuations of the vacuum.

Apart from spontaneous emission, a remarkable phenomenon that follows from the quantization of the field is the Lamb shift. This is a shift in the electronic energy levels of atoms due to vacuum fluctuations. It has been measured for the transition between  $2S_{1/2}$  and  $2P_{1/2}$  energy levels of the hydrogen atom, with the relative accuracy of  $10^{-14}$ !

If initially the atom is in the excited state, and the field has some photons, the atom will emit one more photon and come to the ground state. Since Rabi oscillations are a coherent phenomenon, the resulting photon is in phase with the other photons. This is called **stimulated emission**, and is the basic phenomenon utilized in **lasers**. In addition to the stimulated emission, there will always be spontaneous emission, which causes **fundamental quantum noise limit** in lasers and amplifiers.

If you ever need to know and compare the fully quantum, semiclassical and classical theories of Rabi oscillations and strong coupling, the review P. Törmä and W.L. Barnes, Reports on Progress in Physics 78, 013901 (2015), <http://iopscience.iop.org/article/10.1088/0034-4885/78/1/013901> is recommended. Chapters 1.1-1.3. provide useful background material for this lecture (such as strong coupling of two coupled harmonic oscillators). Chapter 5.3 presents the same derivation as this lecture, and 5.1 gives the corresponding semiclassical description.