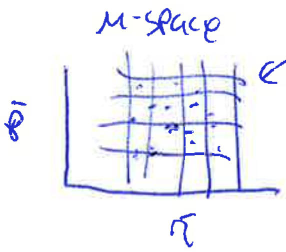
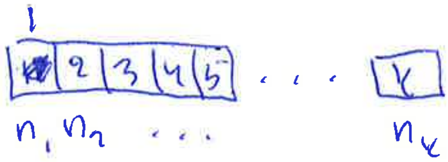
① Points: $\langle n_k \rangle = \dfrac{\sum_{\{n_k\}} n_k \Omega(n_1, n_2 \ldots)}{\sum_{\{n_i\}} \Omega(n_1, n_2 \ldots)}$ vs most probable distribution

$\{\bar{n}_i\}$ ⇒ become the same in the thermodynamic limit





← how many balls in the box?

microstates = number of permutations that leave distribution unchanged

$$\Omega\{n_i\} = \frac{N!}{n_1! n_2! n_3! \ldots} \quad g_1^{n_1} \ldots g_k^{n_k}$$

, g_k = intrinsic probability of the cell i ($g_i = 1$ in the end)

$\Omega\{\bar{n}_i\}$ = maximum ≡ variation in $\{n_i\}$ 2nd order i.e. "small"

⇒ $\delta\Omega\{n_i\} = 0$ when $n_i \to \bar{n}_i + \delta n_i$, so that

$\sum \delta n_i = 0$ number conserved

$\sum \varepsilon \delta n_i = 0$ energy conserved

$\delta \ln \Omega\{n_i\} = 0$ easier to handle (products turn into sums)

$$\delta\left[\ln \Omega\{n_i\} + \alpha \sum_i n_i - \beta \sum \varepsilon_i n_i\right] = 0$$

$$\ln \Omega = \ln N! - \sum_i \ln n_i! + \sum_i n_i \ln g_i$$

Stirling: $\ln N! \approx N \ln N - N$

$$\sum_{n_i} \ln n_i! \approx \sum_i n_i \ln n_i - \underbrace{\sum_i n_i}_{N}$$

$$\Rightarrow \ln \Omega \approx N \ln N - N + \sum_i \left[n_i \ln n_i + (1+\alpha) n_i - \beta \varepsilon_i n_i + n_i \ln g_i \right]$$

$$\to \sum_i \left[-\ln n_i + \alpha - \beta \varepsilon_i + \ln g_i \right] \delta n_i = 0$$

② δn_i are arbitrary so we need

$$\ln n_i = \alpha - \beta\varepsilon_i + \ln g_i \rightarrow n_i = g_i C e^{-\beta\varepsilon_i}, \quad C = e^{\alpha}$$

set $g_i = 1 \rightarrow n_i = C e^{-\beta\varepsilon_i}$ ← Maxwell-Boltzmann distribution.

Number: $C\sum_i e^{-\beta\varepsilon_i} = N$

Energy: $C\sum_i \varepsilon_i e^{-\beta\varepsilon_i} = E$

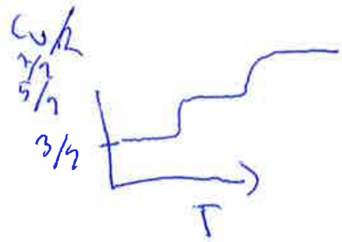
Ideal gas: $\varepsilon_i = p^2/2m$ + replace sums with integrals (over (p_x, p_y, p_z))

$$\Rightarrow C = n\left(\frac{\beta}{2\pi m}\right)^{3/2}, \quad E/N = 3/2\beta$$

Can compute pressure as well: $2mv_x$ momentum change when particle bounces from the wall along x ...

$$p = \frac{2}{3} E/V \quad \& \quad E/N = \frac{3}{2}kT \quad \& \quad \beta = 1/kT.$$

each $\frac{p_x^2}{2m}$ contributes $kT/2$ = equipartition.



Entropy $S = k\sum_{\{n_i\}} \ln \Omega\{n_i\} \approx k \ln \Omega\{\bar{n}_i\}$ ← just keep the largest.

Most probable distribution derivation did not actually assume classical gas.

Probability for <u>state</u> i = $\dfrac{e^{-\varepsilon_i/kT}}{\sum_i e^{-\varepsilon_i/kT}}$

(3) Quantum case: still boxes with n_i particles.
However exchanging particles in different boxes does not produce a new state. Can calculate occupation in each box and multiply

$g_1 \quad g_2 \quad \cdots \quad g_k$ degeneracy

$n_1 \quad n_2 \quad \cdots \quad n_k$

$\Omega\{n_i\} = \prod_j w_j(n_j)$ $\quad w_j$ = number of ways to put n_j into j:th box.

classical was: $\dfrac{N!}{n_1! n_2! \cdots n_k} g_1^{n_1} \cdots g_k^{n_k}$

Fermions: each state either occupied or empty.
n_j into g_j states → pick n_j occupied states among g_j
is binomial coefficient

$$w_j(n_j) = \frac{g_j!}{n_j!(g_j-n_j)!} \rightarrow \Omega = \prod_j \frac{g_j!}{n_j!(g_j-n_j)!}$$

Assume n_j & g_j are large → Stirling approximation

$$\rightarrow \ln\Omega \approx \sum_j \left[g_j \ln g_j - n_j \ln n_j - (g_j-n_j)\ln(g_j-n_j)\right]$$

Most probable: $\delta\left[\ln\Omega - \alpha\sum n_j - \beta\sum \varepsilon n_j\right] = 0$

$$\rightarrow \bar{n}_j = \frac{g_j}{e^{\beta\varepsilon_j - \alpha}+1} \qquad (\alpha = \beta\mu)$$

Bosons: j:th box into g_j compartments with $g_j - 1$ partitions, each compartment can have any number of particles. Vary numbers by moving partitions. Throw n_j in + count number of distinct configurations when permuting

(4) n_j together with $g_j - 1$ partitions

$$w_j(n_j) = \frac{(n_j + g_j - 1)!}{n_j!\,(g_j-1)!}$$

$$\rightarrow \ln \Omega \approx \sum_j \left[(n_j+g_j)\ln(n_j+g_j) - n_j \ln n_j - g_j \ln g_j \right] \quad \left(\begin{array}{c} g_j - 1 \approx g_j \\ \text{also} \\ \text{assumed} \end{array} \right)$$

$$\Rightarrow \ldots \rightarrow \bar{n}_j = \frac{g_j}{e^{\beta \varepsilon_j - \alpha} - 1}$$

Sum over states $\Rightarrow$ sum over k : $\sum_j g_j \rightarrow \sum_k 1$

$$N = \sum_k \frac{1}{e^{\beta \varepsilon_k - \alpha} \pm 1}$$ (cell degeneracy was a mathematical aid)

TDL : $$n = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta \varepsilon_k - \alpha} \pm 1}$$

Must remain finite as $\beta \to 0 \Rightarrow$ need $e^{-\alpha} \xrightarrow[\beta \to 0]{} \infty$

then ± 1 irrelevant. $\beta = 1/kT$ like in Maxwell-Boltzmann

α from fixing n:

$$\rightarrow n\lambda^3 = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^2}{z^{-1} e^{x^2} \pm 1}, \quad z = e^{\alpha} = e^{\beta\mu} = \text{fugacity}$$

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}$$

chemical potential

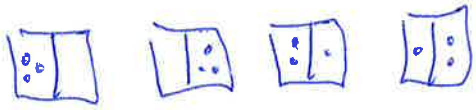
Entropy : $S = k \ln \Omega$ as before

Examples w_j:

Bosons: $w_j(n_j) = \frac{(n_j+g_j-1)!}{n_j!\,(g_j-1)!}$

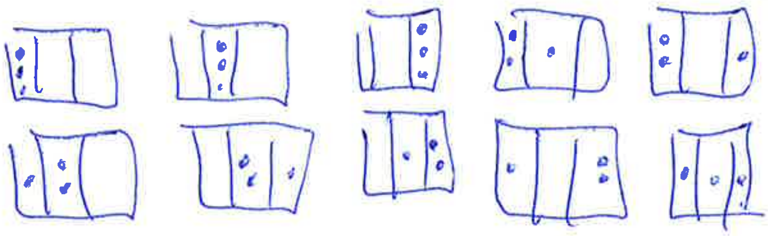
$n_j = 3,\ g_j = 2$

$w_j(3) = \frac{4!}{3!\,1!} = \frac{24}{6} = 4$



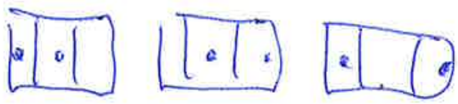
$n_j = 3,\ g_j = 3$

$w_j = \frac{5!}{3!\,2!} = \frac{120}{6\cdot 2} = 10$



Fermions: $w_j = \frac{g_j!}{n_j!\,(g_j-n_j)!}$

$g_j = 3,\ n_j = 2 \Rightarrow w_j = \frac{6}{2\cdot 1} = 3$



Fermions

$$-\delta n_j \ln n_j - \frac{n_j}{n_j}\delta n_j + \delta n_j \ln(g_j - n_j) - \frac{g_j - n_j}{(g_j - n_j)}(-\delta n_j) - \alpha\,\delta n_j - \beta\varepsilon\,\delta n_j = 0$$

$$= \delta n_j\left[-\ln n_j - 1 - \alpha - \beta\varepsilon + 1 + \ln(g_j - n_j)\right] = 0$$

$$\Rightarrow \ln\frac{g_j - n_j}{n_j} = \alpha + \beta\varepsilon \Rightarrow \ln\left(\frac{g_j}{n_j} - 1\right) = \alpha + \beta\varepsilon$$

$$\Rightarrow \frac{g_j}{n_j} = e^{\beta\varepsilon + \alpha} + 1 \Rightarrow \underline{n_j = \frac{g_j}{e^{\beta\varepsilon + \alpha} + 1}}$$

Bosons

$$\ln\Omega = \sum_j (n_j + g_j - 1)\ln(n_j + g_j - 1) - (\cancel{n_j} + \cancel{g_j - 1}) - n_j \ln n_j + \cancel{n_j} - (g_j - 1)\ln(g_j - 1) + \cancel{(g_j - 1)}$$

$$= \sum_j (n_j + g_j - 1)\ln(n_j + g_j - 1) - n_j \ln n_j - (g_j - 1)\ln(g_j - 1)$$

$\Rightarrow$ variation of $\left[\ln\Omega - \alpha\sum n_j - \beta\sum\varepsilon n_j\right]$ should be zero

last term just constant (in $\ln\Omega$).

$$\Rightarrow \sum_j\left[\ln(n_j + g_j - 1) + 1 - \ln n_j - \underset{\nearrow\, -\alpha - \beta\varepsilon}{1}\right]\delta n_j = 0$$

$$\Rightarrow \ln\left(1 + \frac{g_j - 1}{n_j}\right) = \alpha + \beta\varepsilon \Rightarrow \frac{g_j - 1}{n_j} = e^{\alpha + \beta\varepsilon} - 1$$

$$\Rightarrow n_j = \frac{g_j - 1}{e^{\alpha + \beta\varepsilon} - 1}$$

Just checking the algebra here

Distributions via partition function: (Canonical)

1 Total energy in terms of occupations $E_R = \sum_i n_i \varepsilon_i$

Total number $N = \sum_i n_i$

Partition function $Z = \sum_R e^{-\beta E_R} = \sum_R e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 \dots)}$

sum over all configurations $\{n_i\}$

One specific configuration has probability

$$P_R = \frac{e^{-\beta(n_1\varepsilon_1 + \dots)}}{Z}$$ so Z is essentially normalization factor.

mean number $\bar{n}_s = \sum_R n_s P_R = \frac{\sum_R n_s e^{-\beta(n_1\varepsilon_1 \dots)}}{Z}$

$$= -\frac{1}{\beta Z}\frac{\partial Z}{\partial \varepsilon_s} = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_s}$$

key is to find Z.

Maxwell-Boltzmann: each particle has $Z_0 = \sum_r e^{-\beta\varepsilon_r}$

particles distinguishable $\Rightarrow Z = Z_0^N$

$\Rightarrow \ln Z = N \ln \xi = N \ln\left(\sum_r e^{-\beta\varepsilon_r}\right)$.

$\Rightarrow \bar{n}_s = -\frac{1}{\beta} N \frac{(-\beta) e^{-\beta\varepsilon_s}}{\sum_r e^{-\beta\varepsilon_r}} = N \frac{e^{-\beta\varepsilon_s}}{\sum_r e^{-\beta\varepsilon_r}}$ = Maxwell-Boltzmann distribution

or: $Z = \sum_R e^{-\beta(\varepsilon_1 n_1 \dots}$, $n_r = 0, 1, 2 \dots$ for each r, constraint: $\sum n_r = N$

$$= \sum_{n_1, n_2 \dots} \frac{N!}{n_1! n_2! \dots} e^{-\beta(\varepsilon_1 n_1 + \varepsilon_2 n_2 \dots)} = \sum_{n_1, n_2 \dots} \frac{N!}{n_1! n_2! \dots} \left(e^{-\beta\varepsilon_1}\right)^{n_1} \dots$$

since $\sum_r n_r = N$ this is multinomial expansion

6

$$\Rightarrow Z = (e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} \dots)^N \leftarrow \text{same as before}$$

Bosons: n_i can be any integer

$$Z = \sum_r e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 \dots)} = \sum_{n_1, n_2 \dots} e^{-\beta n_1 \varepsilon_1} e^{-\beta n_2 \varepsilon_2} \dots$$

$$= \left(\sum_{n_1=0}^{\infty} (e^{-\beta n_1 \varepsilon_1})\right)\left(\sum_{n_2=0}^{\infty} e^{-\beta n_2 \varepsilon_2}\right) \dots$$

$$\sum_{n_s=0}^{\infty} e^{-\beta n_s \varepsilon_s} = \text{geometric series} = \frac{1}{1-e^{-\beta\varepsilon_s}}$$

$$\Rightarrow Z = \left(\frac{1}{1-e^{-\beta\varepsilon_1}}\right)\left(\frac{1}{1-e^{-\beta\varepsilon_2}}\right) \dots$$

$$\text{so } \ln Z = -\sum_s \ln(1-e^{-\beta\varepsilon_s})$$

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_s} = \frac{e^{-\beta\varepsilon_s}}{1-e^{-\beta\varepsilon_s}} = \frac{1}{e^{\beta\varepsilon_s}-1}$$

Number constraint with $-\mu\sum n_s$ term (Grand canonical Z)

$$\Rightarrow \bar{n}_s = \frac{1}{e^{\beta(\varepsilon_s-\mu)}-1}, \text{ with } \mu \text{ such that } \sum \bar{n}_s = N$$

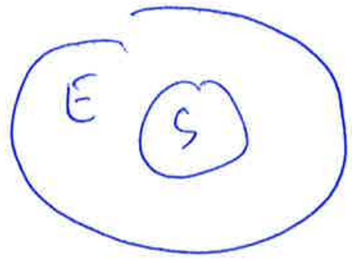
Bose-Einstein distribution

Fermions: $n_s = \{0,1\} \Rightarrow \sum_{n_s=0,1} e^{-\beta n_s \varepsilon_s} = 1 + e^{-\beta\varepsilon_s} \dots \ln Z = \sum_s \ln(1+e^{-\beta\varepsilon_s})$

$$\dots \Rightarrow \bar{n}_s = \frac{1}{e^{\beta(\varepsilon_s-\mu)}+1}$$

why partition function is the way it is?

(7)



total energy is E

p_i = probability for the microstate of S has energy E_i

Assumption: all microstates equally probable

$$\Rightarrow p_i = \frac{\Omega_E(E-E_i)}{\Omega_E(E)}$$
, $\Omega_E(E)$ is the number of microstates for E

Heat bath dominates so $E \gg E_i \Rightarrow$ Taylor expand.

use $\frac{\partial S_E}{\partial E} = 1/T$ (S_E = entropy)

$$k \ln p_i = k \ln \Omega_E(E-E_i) - k \ln \Omega_E(E)$$

$$\approx -\frac{\partial (k \ln \Omega_E(E))}{\partial E} E_i \approx -\frac{\partial S_E}{\partial E} E_i$$

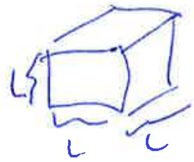
$$\approx -\frac{E_i}{T} \Rightarrow p_i \propto e^{-E_i/kT} = e^{-\beta E_i}$$

Z = normalization

$$= \sum_i e^{-\beta E_i}$$

Note: we did not assume specific statistics here. Just equal probability of microstates. Statistics would matter for how to actually count states, but here that was left "abstract".

BEC derivations: homogeneous space and 3D harmonic trap

Homogeneous space:  $\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z)$, $n_i = 0, \pm 1, \dots$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 (2\pi)^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$$

occupation for some $\vec{n} = (n_x, n_y, n_z)$:

$$n_{\vec{n}} = \frac{z e^{-\beta E_n}}{1 - z e^{-\beta E_n}}, \quad z = e^{\mu\beta} = \text{fugacity} \in [0, 1]$$

$$= \frac{1}{e^{+\beta(E_n - \mu)} - 1}$$

sum these to get total number. $E_n = 0$ could be special since $n_{\vec{n}} \to \infty$ if $\mu = 0$ as well.

$$\Rightarrow \frac{N}{L^3} = \frac{1}{L^3}\sum_{\vec{n}} n_{\vec{n}} = \frac{z}{(1-z)L^3} + \frac{1}{L^3}\sum_{\vec{n} \neq 0} \frac{z e^{-\beta E_n}}{1 - z e^{-\beta E_n}}$$

Replace sum with integral (large N, lots of levels E_n involved)

2nd term: $\frac{1}{L^3}\int_{-\infty}^{\infty} dn_x \int_{-\infty}^{\infty} dn_y \int_{-\infty}^{\infty} dn_z \frac{1}{e^{\beta(E_n - \mu)} - 1}$ (*)

spherical coordinates: $r^2 = \underbrace{\frac{\beta \hbar^2 (2\pi)^2}{2mL^2}}_{A \text{ = dimensionless}} (n_x^2 + n_y^2 + n_z^2)$

$$\begin{cases} n_x = r/\sqrt{A} \sin\theta \cos\varphi \\ n_y = r/\sqrt{A} \sin\theta \sin\varphi \\ n_z = r/\sqrt{A} \cos\theta \end{cases}$$

$$\Rightarrow \iiint dn_x dn_y dn_z = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} dr \frac{r^2}{(A)^{3/2}} \sin\theta$$

angle integrals give 4π

$$(*) = \frac{4\pi}{L^3 A^{3/2}} \int_0^\infty dr \frac{r^2}{\frac{1}{z}e^{r^2} - 1} = \frac{4\pi}{L^3 A^{3/2}} \int_0^\infty dr \frac{z e^{-r^2} r^2}{1 - z e^{-r^2}}$$

prefactor: $\frac{4\pi(2m)^{3/2}L^3}{L^3\beta^{3/2}(2\pi)^3\hbar^3} = \frac{1}{2\pi^2}\cdot\frac{2\sqrt{2}}{\hbar^3}\left(\frac{m}{\beta}\right)^{3/2} = \frac{\sqrt{2}}{\pi^2}\left(\frac{\sqrt{mk_BT}}{\hbar}\right)^3$

$$= \frac{\sqrt{2}}{\pi^2}\left(\frac{\sqrt{2\pi}}{\sqrt{\frac{\hbar^2\cdot 2\pi}{mk_BT}}}\right)^3 = \left(\frac{1}{\lambda_{dB}}\right)^3\cdot\frac{\sqrt{2}}{\pi^2}(2\pi)\sqrt{2}\sqrt{\pi}$$

$$= \left(\frac{1}{\lambda_{dB}}\right)^3\frac{4}{\sqrt{\pi}}$$

$$\Rightarrow (*) = \left(\frac{1}{\lambda_{dB}}\right)^3\frac{4}{\sqrt{\pi}}\int_0^\infty d\eta\left(\frac{z e^{-\eta^2}\eta^2}{1-ze^{-\eta^2}}\right) = \left(\frac{1}{\lambda_{dB}}\right)^3\cdot g_{3/2}(z)$$

Riemann zeta function: $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^s}{e^x-1}\frac{dx}{x}$

Gamma function: $\Gamma(s) = \int_0^\infty e^{-x}x^s\frac{dx}{x}$

If $\frac{4}{\sqrt{\pi}}\int_0^\infty d\eta\,\eta^2\frac{ze^{-\eta^2}}{1-ze^{-\eta^2}} = g_{3/2}(z)$, $x=\eta^2$, $d\eta = \frac{1}{2\sqrt{x}}dx$

$$\Rightarrow \frac{4}{\sqrt{\pi}}\frac{1}{2}\int_0^\infty\frac{dx}{x}z\left(\frac{x^{3/2}e^{-x}}{1-ze^{-x}}\right) = \frac{2}{\sqrt{\pi}}\int_0^\infty\frac{dx}{x}\frac{x^{3/2}}{\frac{1}{z}e^x-1}$$

Note: if $z=1$, we see $\zeta(3/2)$ in the integral $\times\Gamma(3/2)$

$\Gamma(3/2) = \sqrt{\pi}/2$ so ...

So we have the $\frac{1}{L^3}\sum_{\vec{n}\neq 0}\frac{ze^{-\beta E_n}}{1-ze^{-\beta E_n}} \xrightarrow{z\to 1} \left(\frac{1}{\lambda_{dB}}\right)^3\zeta(3/2)$

$\zeta(3/2)\approx 2.612$

(For other z $g_{3/2}(z)$ is related to polylogarithm $\mathrm{Li}_{3/2}(z)$ ... we don't care)

Important to note that $g_{3/2}(z)$ increases monotonically with z so that maximum is at $z=1$ $\Rightarrow$ Maximum number in the sum

$$\max\left(\frac{1}{L^3}\sum_{\vec{n}\neq 0}\frac{z e^{-\beta E_n}}{1-z e^{-\beta E_n}}\right) = \max\left(\frac{1}{\lambda_{dB}^3} g_{3/2}(z)\right)$$

$$= \frac{1}{\lambda_{dB}^3} g_{3/2}(1) = \left(\frac{1}{\lambda_{dB}}\right)^3 \zeta(3/2) \approx 2.612/\lambda_{dB}^3$$

Thermal density bounded from above!
If density larger than this, excess must go to the state with $(n_x, n_y, n_z) = (0,0,0)$ i.e. the ground state.

Critical temperature from $\mu = E_0 = 0$ and requiring density from thermal part to be the same as total density.

$$\rightarrow n = \frac{\zeta(3/2)}{(\lambda_{dB})^3} \Rightarrow \frac{n}{\zeta(3/2)} = \left(\frac{m k_B T_c}{2\pi\hbar^2}\right)^{3/2}$$

$$\Rightarrow T_c = \left(\frac{2\pi\hbar^2}{m k_B}\right)\left(\frac{n}{\zeta(3/2)}\right)^{2/3},$$

$^{87}Rb \rightarrow m = 1.44 \cdot 10^{-25}$ kg
$n = 10^{14}$ /cm^3

$\rightarrow T_c \approx 400$ nK

BEC more generally:

We always sum over excited states. Move from that sum into integral over energy $\Rightarrow$ need density of states.

Free particle: $\varepsilon_p = p^2/2m \rightarrow p = \sqrt{2m\varepsilon_p}$

volume of sphere with radius p: $\frac{4\pi}{3}(2m\varepsilon)^{3/2}$

one quantum state per volume $(2\pi\hbar)^3$ of phase space

$\Rightarrow$ number of states with energy less than ε:

$$G(\varepsilon) = V\,\frac{4\pi}{3}\frac{(2m\varepsilon)^{3/2}}{(2\pi\hbar)^3} \qquad V = \text{system volume}$$

density of states $g(\varepsilon) = \frac{dG(\varepsilon)}{d\varepsilon} = \frac{V m^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}\sqrt{\varepsilon}$

This was in 3D. Constant in 2D and $\propto 1/\sqrt{\varepsilon}$ in 1D.

3D harmonic oscillator:

$$\varepsilon(n_x,n_y,n_z) = (n_x+1/2)\hbar\omega_x + (n_y+1/2)\hbar\omega_y + (n_z+1/2)\hbar\omega_z$$

when ε large compared to $\hbar\omega_\alpha$ we can treat n_α as continuous. 3D space with variables $\varepsilon_\alpha = \hbar\omega_\alpha n_\alpha$ $\alpha \in \{x,y,z\}$

Constant energy surface $\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$ (ignore ground state energies as small here). $G(\varepsilon)$ = volume in the first octant bounded by this surface.

$$G(\varepsilon) = \frac{1}{\hbar^3\omega_x\omega_y\omega_z}\int_0^{\varepsilon} d\varepsilon_x \int_0^{\varepsilon-\varepsilon_x} d\varepsilon_y \int_0^{\varepsilon-\varepsilon_x-\varepsilon_y} d\varepsilon_z = \varepsilon^3/(6\hbar^3\omega_x\omega_y\omega_z)$$

$$\Rightarrow g(\varepsilon) = \frac{\varepsilon^2}{2\hbar^3\omega_x\omega_y\omega_z}$$

For d-dimensional gas $g(\varepsilon) = \frac{\varepsilon^{d-1}}{(d-1)!\prod_{i=1}^{d}\hbar\omega_i}$

often $g(\varepsilon) \propto \varepsilon^{\alpha-1}$ structure. Assume $g(\varepsilon) = C_\alpha \varepsilon^{\alpha-1}$

Number of excited atoms:

$$N_{ex} = \int_0^\infty d\varepsilon \, g(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$

T_c from $N_{ex}(T_c, \mu=0) = N = \int_0^\infty d\varepsilon \, g(\varepsilon) \frac{1}{e^{\varepsilon\beta}-1}$

$x = (\varepsilon / k_B T)$, $d\varepsilon = k_B T \, dx$, $x \in [0, \infty]$

$$\Rightarrow N = C_\alpha (k_B T_c)^\alpha \int_0^\infty dx \frac{x^{\alpha-1}}{e^x - 1} = C_\alpha \Gamma(\alpha) \zeta(\alpha) (k T_c)^\alpha$$

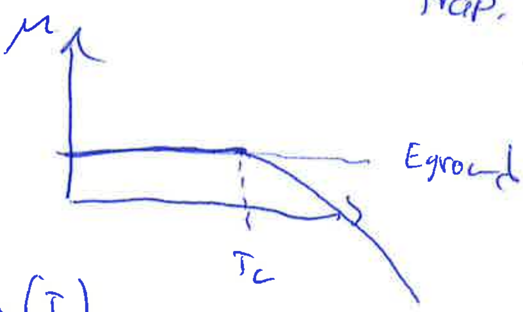
Gamma function   Riemann zeta.

3D harmonic oscillator:

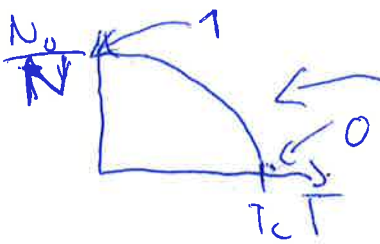
$$k T_c = \frac{\hbar \bar{\omega} N^{1/3}}{\zeta(3)^{1/3}}, \quad \bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$$

If $\alpha = 1$: $\zeta(\alpha = 1) = \infty$ so T_c goes to zero. NO BEC. This is the case in 2D without the trap and in 1D harmonic oscillator. (BEC is possible in 2D harmonic ~~trap~~ trap.)

Below T_c: $\mu = E_{ground}$ = often zero.



$$N_{ex} = \int_0^\infty d\varepsilon \, g(\varepsilon) \frac{1}{e^{\beta\varepsilon}-1} = N_{ex}(T)$$



$\rightarrow N_0 = N - N_{ex}(T)$

condensate number

functional form depends on density of states.

Mean field theory and Gross-Pitaevski equation

Hamiltonian $H = \int d^3x\, \hat{\Psi}^+(x)\left[\frac{p^2}{2m} + V_T(\bar{x})\right]\hat{\Psi}(x) + \frac{1}{2}\frac{4\pi\hbar^2 a_s}{m}\int d^3x\, \Psi^+(x)\Psi^+(x)\Psi(x)\Psi(x)$

Here interactions between particles: $V(x-x') = \frac{4\pi\hbar^2}{m} a_s\, \delta(x-x')$ why works?

Grand potential in condensed state $\langle\phi|\hat{H} - \mu\hat{N}|\phi\rangle$

Non-interacting $\Rightarrow$ N bosons in the lowest $\tilde{\varphi}_0(\bar{x})$ state of

$$\left[\frac{p^2}{2m} + V_T(x)\right]\tilde{\varphi}_n(\bar{x}) = \varepsilon_n \tilde{\varphi}_n(\bar{x})$$

$$|\phi\rangle = \frac{(a_0^+)^N}{\sqrt{N!}}|\text{vacuum}\rangle, \quad \hat{a}_0^+ = \int d^3x\, \tilde{\varphi}_0(x)\hat{\Psi}^+(x)$$

With interactions? $\hat{\Psi}(x) = \sum \hat{a}_n \varphi_n(x)$ Not yet known

Pure condensate $|\phi\rangle = \frac{(\hat{a}_0^+)^N}{\sqrt{N!}}|\text{vacuum}\rangle = |N, 0, 0 \cdots\rangle$

$$\langle\phi|H - \mu N|\phi\rangle = N\int d^3x\, \underbrace{\varphi_0^*(x)\left[\frac{p^2}{2m} + V_T(x) + (N-1)\frac{2\pi\hbar^2 a_s}{m}|\varphi_0|^2 - \mu\right]\varphi_0(x)}_{\mathcal{L}(\bar{x})\, =\, \text{"Lagrangian density" prob 9.1}}$$

Euler-Lagrange equation $\Rightarrow$ Gross-Pitaevskii equation for the condensate wavefunction Prob 9.2

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_T(x) + N\frac{4\pi\hbar^2 a_s}{m}|\varphi_0(x)|^2\right)\varphi_0(x) = \mu\varphi_0$$

For large number of atoms interaction term and potential energy term often large compared to kinetic energy term $\propto \nabla^2 \Rightarrow$ Thomas-Fermi approximation.
Prob. 9.3.

This approach ok when condensate is almost pure and dilute i.e $a_s^3 \cdot n \ll 1$

Typically $a_s \sim 100 a_0$ (a_0: Bohr radius) $\Rightarrow n \ll 10^{17}\ 1/cm^3$ ($10^{14} - 10^{15}$ in experiments)

Note that for air $n_{air} \approx 2 \cdot 10^{19}\ 1/cm^3$

Condensate wavefunction / manybody wavefunction

$\varphi_0(x)$ is a wavefunction of what exactly? electrons, outer electron, nucleus... something else?

Hydrogen: $\Psi(X_1, X_2)$ X_1 = pos. of proton
X_2 = pos of electron

$V(X_1, X_2) = V(|X_1 - X_2|)$

$\Rightarrow \bar{R} = \dfrac{m_p \bar{X}_1 + m_e X_2}{m_p + m_e}, \quad \bar{r} = \bar{X}_2 - \bar{X}_1$

$\Psi(X_1, X_2) \to \Psi(\bar{R})\varphi(\bar{r})$ ← normally this one is solved in text books!

$e^{i\bar{K}\cdot\bar{R}}, \; E = \dfrac{\hbar^2 K^2}{2(m_e + m_p)}$

not super interesting

However, the CM wavefunction is the relevant for example thermal distribution of atoms. Atoms have (almost) all the same electronic part of the wf. It is the center of mass part that are different. BEC wavefunction is for the center of mass coordinate!

Lagrangian density thing bit more carefully:

Actually in QM Lagrangian density is

$$\mathcal{L} = \frac{i\hbar}{2}\left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right) - H[\Psi, \Psi^*],$$

note time derivatives. Also, classically $L = \frac{m}{2}\dot{x}^2 - V(x)$

Schrödinger eq:

$$\frac{\partial \mathcal{L}}{\partial \Psi^*} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^*}{\partial t}} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^*}{\partial x_j}} = 0 \Rightarrow i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

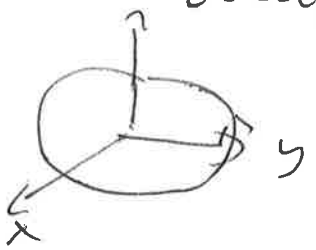
Then $\Psi \to \Psi e^{-\mu t/\hbar} \Rightarrow LHS = \mu\Psi$

You get what is in the notes by making the above replacement in the earlier Lagrangian.

1/2 Vortices:

Solid body rotation $\bar{v} = \hat{\Omega} \times \bar{r}$

$\bar{r} = (x, y, 0)$



$\Rightarrow \bar{v} = -y\Omega\hat{e}_x + x\Omega\hat{e}_y$

vorticity: $\nabla \times \bar{v} = 2\Omega\hat{e}_z$

circulation $\int (\nabla \times \bar{v}) \cdot d\bar{s} = 2\pi r^2 \Omega$

or ~~$\oint \bar{v} \cdot dl$~~ $\oint \bar{v} \cdot dl = \int (-r\sin\phi\,\Omega, r\cos\phi\,\Omega, 0)(-r\sin\phi, r\cos\phi, 0)\,d\phi$

line integral around ring

$= \int_0^{2\pi} r^2 \Omega\, d\phi = 2\pi r^2 \Omega$ = same as before.

Quantum case: $\Psi = \sqrt{\rho}\, e^{iS}$, $\hat{p} = -i\hbar\nabla$

$\Rightarrow \hat{p}\Psi = -i\hbar(\nabla\rho + \underbrace{i\sqrt{\rho}\nabla S}_{\text{interesting part}})e^{iS}$ "$= m\bar{v}\Psi$"

$\Rightarrow$ identify $\bar{v} = \frac{\hbar}{m}\nabla S$ (check units $\frac{kg\,m^2}{s\,kg\,m} = \frac{m}{s}$ = ok)

Phase funktion kind of like a potential for $\bar{v}$-field.

$\nabla \times \bar{v} \propto \nabla \times \nabla S = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ \partial_x S & \partial_y S & \partial_z S \end{vmatrix}$ ignore z for simplicity

$= \hat{e}_x \cdot 0 + \hat{e}_y \cdot 0 + \hat{e}_z(\partial_x\partial_y S - \partial_y\partial_x S) = 0$??

$\Rightarrow$ circulation $\int \nabla \times \bar{v} \cdot d\bar{s} = 0$ as well!!

How can a system described by a wavefunction rotate ??