Lecture 9 Fermi liquid theory

Literature: T. Giamarchi, Quantum Physics in One Dimension, Oxford Science Publications, Chapter 1.1., G.D. Mahan, Many-Particle Physics, Kluwer, Chapter 11.2.1, E.M. Lifshitz and L.P. Pitaevskii, Landau and Lifshitz Course of Theoretical Physics, Statistical Physics Part 2, Chapters 1 and 2

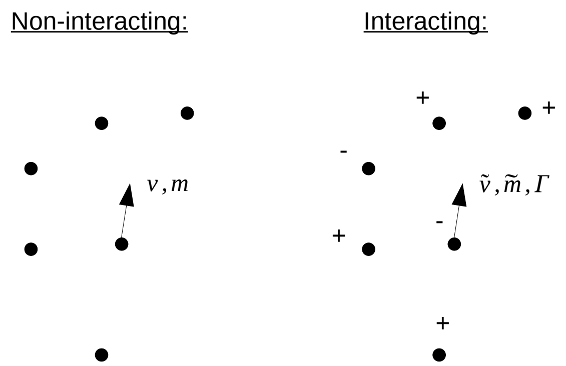
Learning goals

- To learn the concept of a quasi-particle, to understand why there can be well-defined quasiparticles close to the Fermi level.
- To learn what is a Fermi liquid.
- Fermi liquid theory will be continued in Lecture 10. So you can achieve the above learning goals only after that lecture.

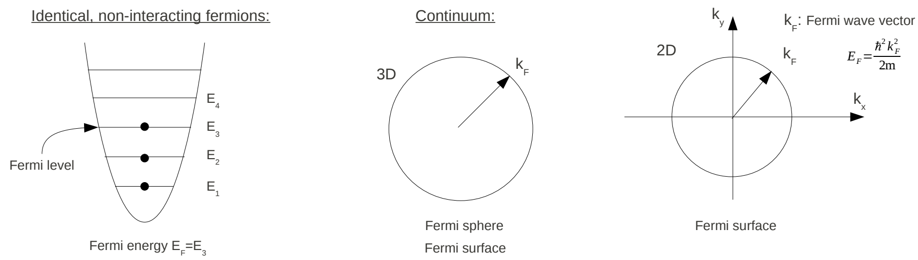
17 Basic concepts for Fermions

Describing interacting Fermions, in real-life situations, is often difficult. For instance for an electron gas (the conduction electrons in a metal), the Coulomb interactions are of long-range type, and they can be in magnitude of the same order as the kinetic energy: in such a situation, one cannot do perturbation theory using the interaction (or kinetic) energy as the small quantity. However, Lev Landau suggested in 1957 a simple but powerful theory to describe interacting Fermions. It is called Fermi liquid theory (sometimes Landau's Fermi liquid theory). It is based on the assumption that elementary low-energy excitations of the systems can be described as so-called **quasiparticles**. Important properties such as thermodynamic quantities can then be calculated by considering the quasiparticles and simple interactions between them. You have also perhaps learned the free electron model in some previous lecture course. Why can one actually treat electrons as free even when they interact pretty strongly with other electrons via the Coulomb interaction? In this lecture we will learn the basics of why it is actually reasonable to use the concept of a quasiparticle to describe, e.g., metals, and based on the quasiparticle concept, the approximation of a free electron becomes more plausible as well.

Quasiparticles are not the same as the actual particles of the system, except for the non-interacting case where the quasiparticle reduces to the actual particle. One can imagine quasiparticle motion to be, for instance, something like the motion of an electron influenced by the electrons around it. For instance, the electron could move slower due to the Coulomb interaction with the other electrons, and thus one could describe it as a quasiparticle with a bigger effective mass. One can say that the particle is "dressed" by the other particles around it. Furthermore, the quasiparticle typically has a finite lifetime τ (in other words, a finite decay rate Γ); a usual particle in a non-interacting system will maintain its momentum indefinitely, but a quasiparticle excitation of a certain momentum will eventually decay. However, the lifetime has to be long enough in order to the quasi-particle to be well-defined. The figure below illustrates the concept.

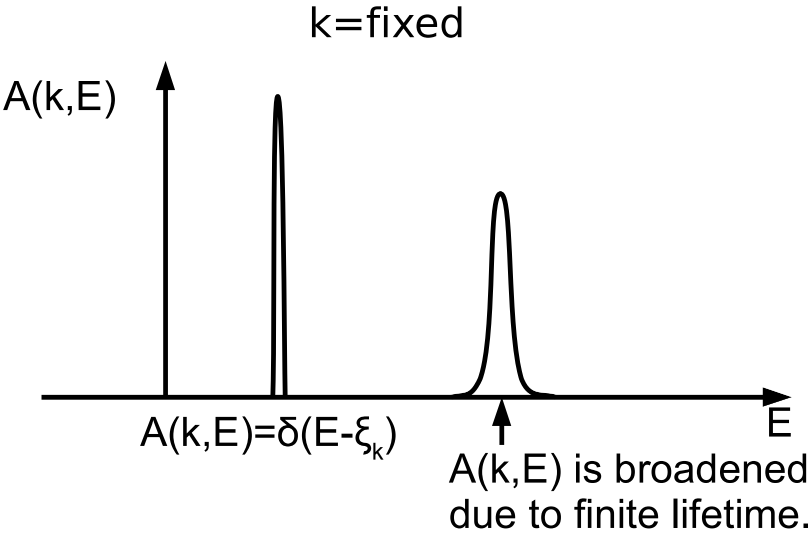


Note that this is a just a vague hand-waving picture of what a quasiparticle is. The more exact definition will be given below. But first we have to learn some basics about non-interacting Fermions.
Important concepts:
Fermi level, Fermi sea, Fermi sphere, Fermi energy, Fermi wave vector (explained in the pictures below).



The chemical potential μ in case of fermionic particles is also called the Fermi level. At zero temperature and for non-interacting particles, it coincides with the Fermi energy E_F which is the highest energy level occupied by the non-interacting particles at $T = 0$, as depicted in the above picture. In presense of interactions and at finite temperature the Fermi level deviates from this. It still tells, in an overall manner, about the energies that are occupied by the fermions, but for instance at finite temperature there is some population also at states above the Fermi level. In general, μ deviates from E_F due to interactions and temperature, although in practise they can be close to each other in some systems.

We should also learn the concept of a spectral function $A(\mathbf{k}, E)$. It tells about the available states in the system for each value of the momentum $\mathbf{k}$ and the energy E. For non-interacting particles it is simply $A(\mathbf{k}, E) = \delta(E - \xi(\mathbf{k}))$ where $\xi(\mathbf{k}) = \epsilon(\mathbf{k}) - \mu$ and $\epsilon(\mathbf{k})$ is the kinetic energy, for instance $\epsilon(\mathbf{k}) = \hbar^2 k^2/(2m)$. For the non-interacting case, the spectral function is thus extremely sharp, see the picture below. In contrast, for a state of the system that has a finite lifetime, the spectral function is broadened, see below.



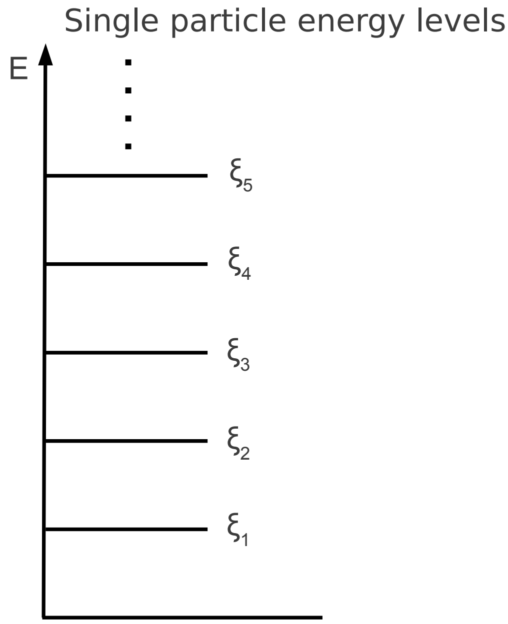
Now, let us proceed towards the Fermi liquid.

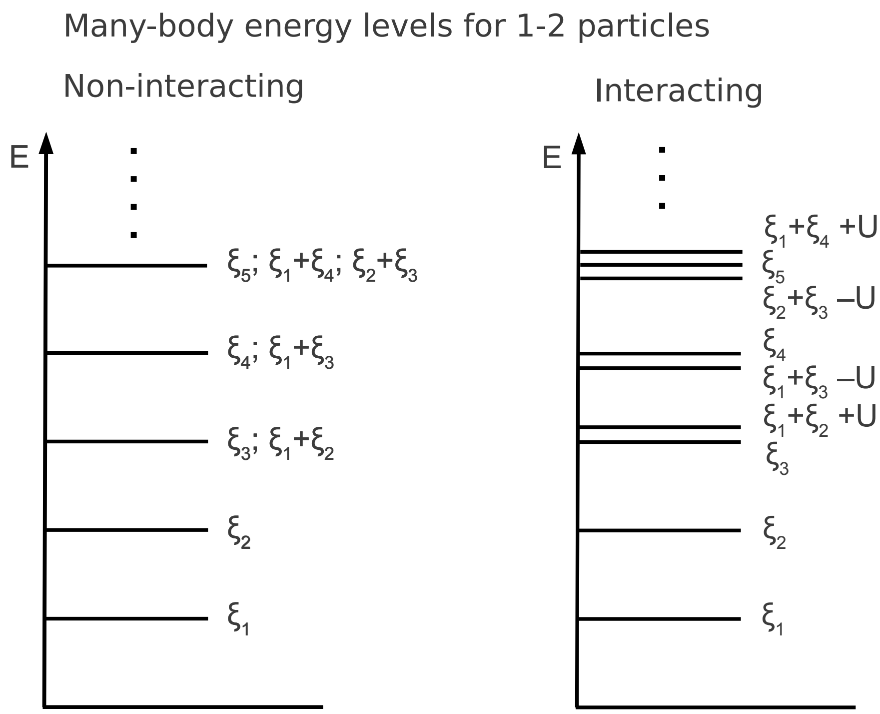
18 A simple example as an introduction to the Fermi liquid

Let us look at a minisystem of one or two fermions that can occupy different momentum states in one dimension, and let us state that they interact depending on which momentum state they are, in the following way:

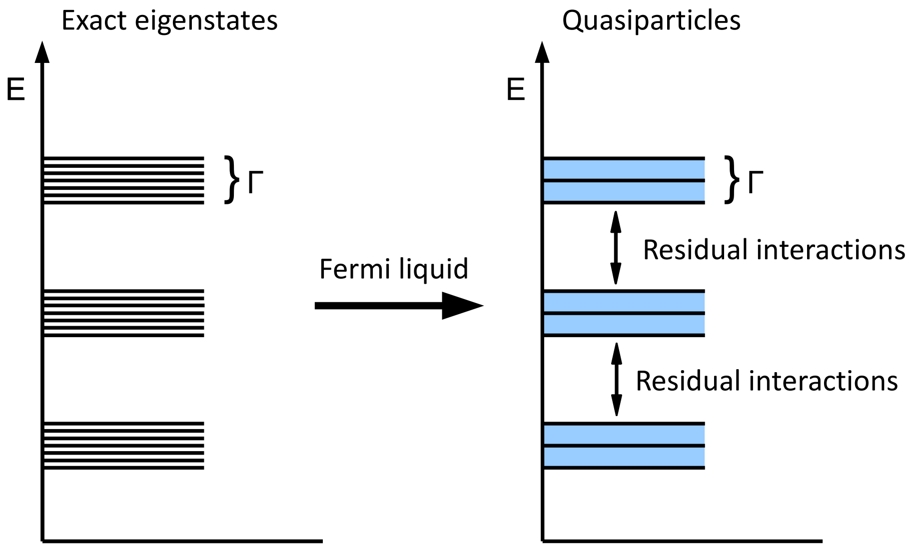
$$H = \sum_k \xi_k c_k^\dagger c_k + U c_1^\dagger c_2^\dagger c_2 c_1 - U c_1^\dagger c_3^\dagger c_3 c_1 + U c_1^\dagger c_4^\dagger c_4 c_1 - U c_2^\dagger c_3^\dagger c_3 c_2. \tag{18.1}$$

In the below picture, the energies corresponding to different choices of momenta for the two particles are given for the non-interacting and interacting case. In the non-interacting case, the energies for the two-particle case can be given as simple sums of the single particle energies. Note that different choices of momenta for the two particles, for instance 1 and 4 or 2 and 3, may give the same total energy: the many-particle energy levels are degenerate. You may imagine that such a degeneracy can be much bigger for a system with more particles and in a higher dimension where several choices of direction of momentum can lead to the same energy. In the interacting case, the degeneracy is broken. In our toy example, the interaction energy is U for the momentum choice 1 and 4, and $-U$ for the choice 2 and 3: this splits the corresponding many-body energy level. Such lifting of degeneracy happens in real-world large systems as well, whenever the interaction energy has a non-trivial momentum dependence. And it usually does: the interaction energy in position space typically has a certain non-trivial form, for instance it is a contact interaction, or the Coulomb interaction with $1/r$ decay, or an interaction potential with a minimum like the Van der Waals interaction. Consequently, the Fourier transform of the interaction energy, that is, the momentum dependence of the interaction, will also be non-trivial. This leads to the splitting of the degenerate non-interacting many-body energy levels into a multitude of different new eigenstates of the interacting many-body system.





The exact eigenstates of an interacting many-body state can be always found, in principle (in practise often not!), but there are in general exponentially many of them. The idea behind quasiparticles in the Fermi liquid theory is to assume that the large number of many-body eigenstates can be lumped around the non-interacting energies a bit like in our example above. Then these lumps are effectively forming the quasiparticles, see the picture below, thus, the number of quasiparticles will not be exponentially large. Sometimes the formation of a Fermi liquid is described in the following way: one starts from the non-interacting case, and adiabatically turns on the interactions. In this process the non-interacting states deform into the quasiparticle states. Our example and the pictures above and below are helpful in imagining what this may mean.



The set of exact eigenstates around the non-interacting energy is assumed to approximately form a band that gives the quasiparticle decay rate Γ (finite lifetime τ). In other words, the effect of interactions will be expressed as the finite width of the spectral function. Furthermore, the quasiparticle may have other properties that deviate from the non-interacting particle, like the effective mass. And, importantly, the quasiparticles are assumed to (weakly) interact with each other: these residual interactions give the so-called **Landau parameters** which determine for instance the thermodynamics of the system.

In Lecture 10, we will proceed to discuss the Fermi liquid theory more rigorously.

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19 Scattering close to the Fermi level and the quasiparticle lifetime

We are now convinced that entities like quasiparticles might exist. Let us say they have an energy $E(k)$; at the moment we do not know its value. Close to the Fermi surface, one can linearize this dispersion in powers of $k - k_F$ (here we use the notation $|\mathbf{k} - \mathbf{k}_F| \equiv k - k_F$)

$$E(k) \simeq E(k_F) + \frac{\hbar^2 k_F}{m^*}(k - k_F). \tag{19.1}$$

This is how the effective mass m^* of the quasiparticle is defined. One can also immediately check that for the non-interacting system this becomes (note that E(k) includes the chemical potential, that is, $E(k_F) = \hbar^2 k_F^2/2m - \mu = 0$)

$$E(k) \simeq \frac{\hbar^2 k_F}{m}(k - k_F). \tag{19.2}$$

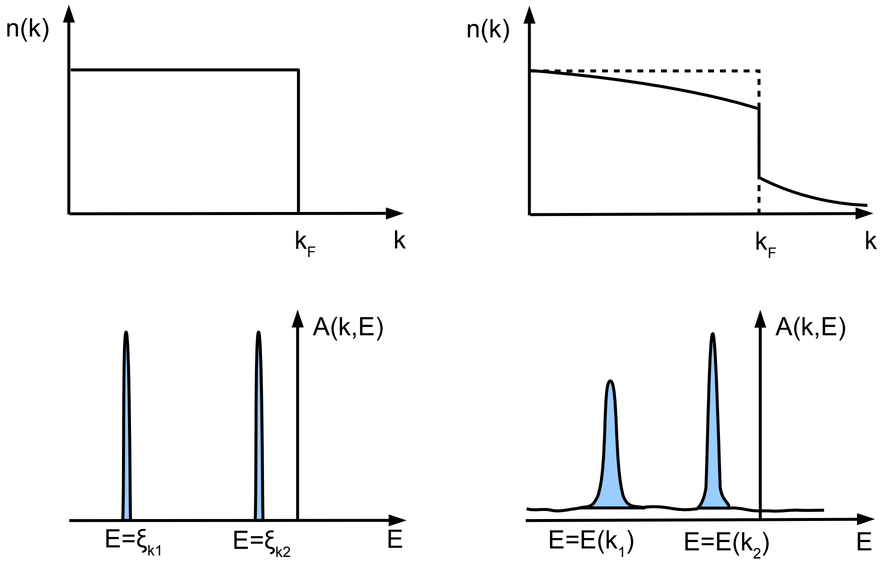
As we already learned, the quasiparticles must have a finite life-time τ, due to the interactions. The wavefunction of the quasiparticle with energy $E(k)$ and lifetime τ would evolve according to

$$e^{-iE(k)t/\hbar}e^{-t/\tau}. \tag{19.3}$$

Such a time-dependence corresponds to a Lorentzian lineshape for the spectral function $A(k, \omega)$. In case the lifetime τ was a constant, the system would become overdamped when approaching the Fermi level, i.e. the damping time τ would be smaller than the period $\hbar/E(k)$ since $E(k)$ approaches zero close to the Fermi level. However, τ in general is not a constant but depends on the interactions, that is, on the scatterings between the particles. Well above the Fermi surface, the particles can scatter freely. **But close to the Fermi level, the phase space for scattering is limited since the states below the Fermi level are already occupied. Landau has shown that in three dimensions, the lifetime diverges as $\tau \sim 1/E(k)^2$.** Since the period of the wavefunction oscillation diverges only as $\sim 1/E(k)$, the growth of the lifetime is always stronger, and the quasiparticles at the Fermi level become very long-lived compared to the inverse of their energy. **In other words, the quasiparticles close to the Fermi level are indeed well-defined quasiparticles. This non-availability of phase space for scattering in a Fermi system is basically the reason why the simple Fermi liquid theory with quasiparticles is often a reasonably good description of the complex system.** Actually, the lifetimes close to the Fermi surface are so long that the damping is often neglected and quasiparticle spectral functions are approximated by delta-functions. The free electron model that you may have learned previously is hopefully more intuitive now!

This discussion has also a further implication: the concept of a quasiparticle is *not* good when one is too far away from the Fermi level. Temperature has the effect of broadening the Fermi level: the particles are located in energy in a region around the Fermi surface that is characterized by $k_B T$. The phase space argument of scattering, based on Pauli blocking, is partly removed from this area. Thus for the quasiparticles to be well-defined (long-lived), one should have the temperature smaller than the Fermi energy. For metals, the Fermi energy is about 10 000 K, so one can be sure that that quasiparticles at room temperature (of the order 100 K) are well defined. The Fermi liquid theory sometimes works also when the temperature difference is not so huge.

The spectral functions of the non-interacting (left) and the interacting (right) system are shown in the figure below. While in the former the spectral functions are delta-functions, in the latter the width of the spectral function is finite but approaches zero when going closer to the Fermi energy.



Also the momentum distributions $n(k)$ are shown in the above figure. For a non-interacting system, the momentum states are filled up to the Fermi level. Then there is a sharp drop in the momentum distribution. The existence of such a drop is equivalent to saying that there exists a Fermi surface. For an interacting system in a Fermi liquid state, there are still well defined, long-lived quasiparticle excitations close to the Fermi surface: therefore one sees a sharp drop in the momentum distribution. However, the drop is smaller than in the non-interacting case: the smoothening of the Fermi surface is due to the fact that not all of the liquid can be described as quasiparticles, only the area close to the Fermi energy. The proportion of the "sharp drop" compared to the size of the drop in the non-interacting case (which is one, obviously) is called the quasiparticle weight and often denoted Z. It tells how big proportion of the liquid can actually thought to be described as free quasiparticles.

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