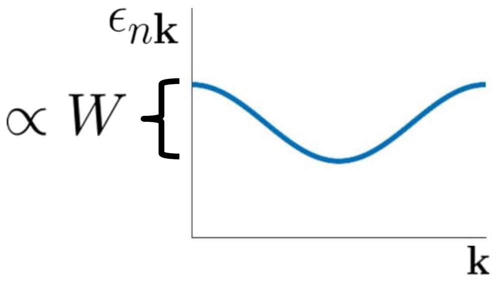
Flat bands: interactions dominate

Dispersive band U<<W:



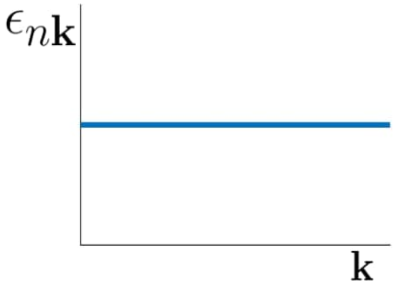
$$\psi_n(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n\mathbf{k}}(\mathbf{r})$$

(periodic part of) the Bloch function

T_c for Cooper pairing

$$T_c \propto e^{-1/(U n_0(E_f))}$$

Flat band U>>W:



$\epsilon_{n\mathbf{k}} =$ constant

Group velocity: $\frac{\partial \epsilon_{n\mathbf{k}}}{\partial k} = 0$

No interactions: insulator at any filling

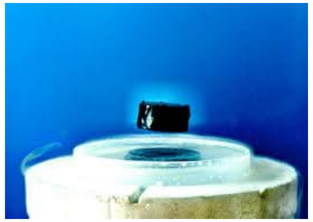
$$T_c \propto U V_{\mathrm{flat\ band}}$$

High T_c for pairing
(Khodel, Shaginyan, Volovik, Kopnin, Heikkilä)

This is the critical temperature for Cooper pairing

$$\Delta(\mathbf{r}) = \langle \psi_\sigma(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) \rangle \qquad \Delta(\mathbf{r}) = |\Delta(\mathbf{r})|$$

Superfluid weight: supercurrent and Meissner Effect



Supercurrent

$$\mathbf{j} = -D_s \mathbf{A}$$

Current

$$\mathbf{j} = \sigma \mathbf{E} \qquad \mathbf{E} = -\partial \mathbf{A}/\partial t$$

Order parameter phase gradient $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})| \mathrm{e}^{2i\phi(\mathbf{r})}$

$\nabla \phi - e\mathbf{A}/\hbar$ Invariant under gauge transformations

Free energy change associated with phase gradient

$$\Delta F = \frac{\hbar^2}{2e^2} \int \mathrm{d}^3\mathbf{r} \sum_{ij} [D_s]_{ij} \partial_i \phi(\mathbf{r}) \partial_j \phi(\mathbf{r})$$

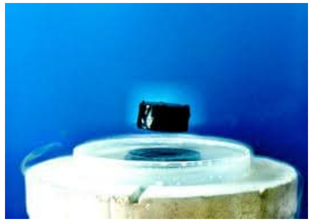
London equation and penetration depth

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\nabla^2 \mathbf{B} = \mu_0 D_s \mathbf{B}$$

$$\lambda_L = (\mu_0 D_s)^{-1/2}$$

Superfluid weight:
supercurrent and Meissner Effect



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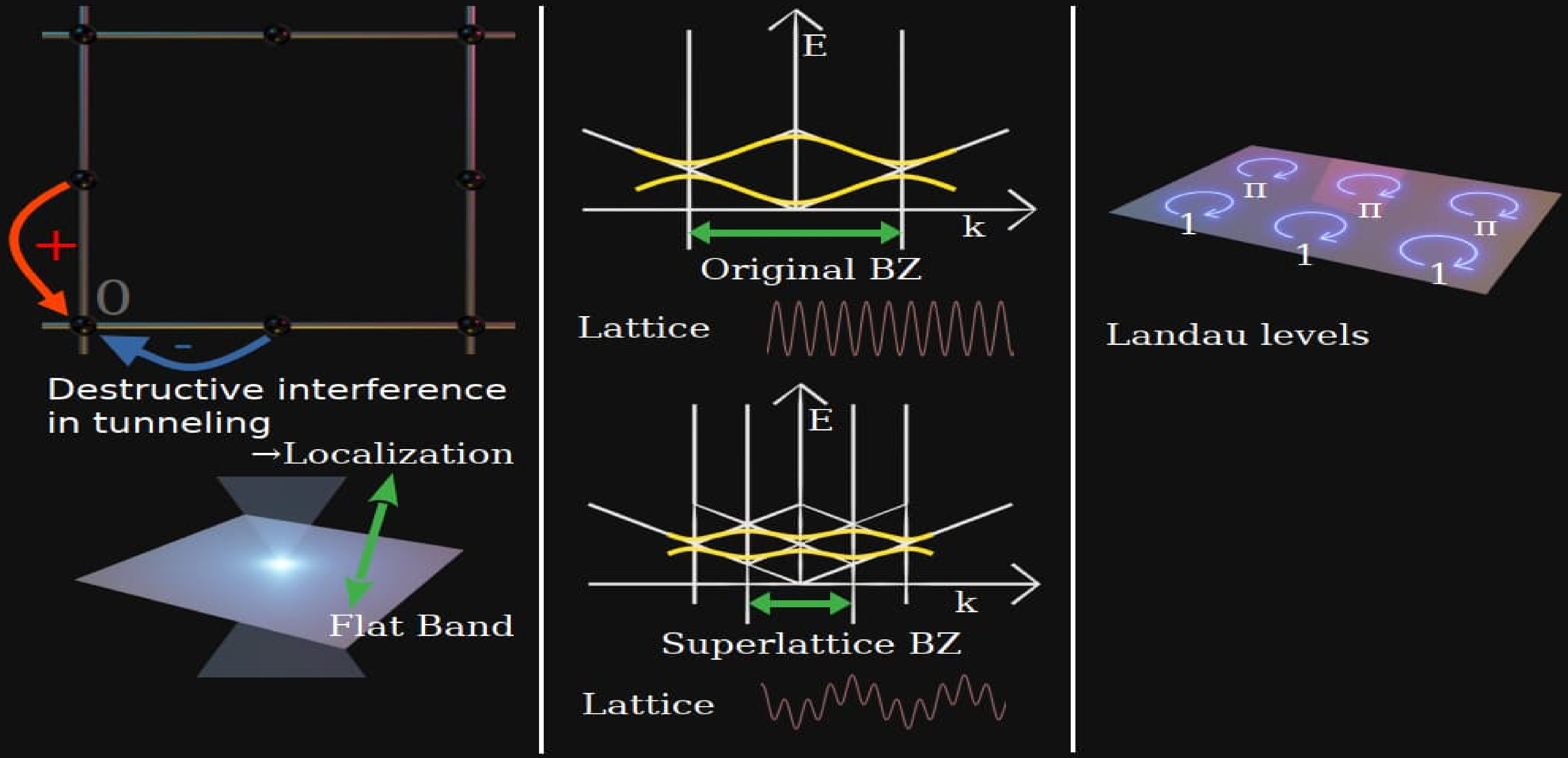
Conventional BCS: $D_s = \frac{e^2 n_\mathrm{p}}{m_\mathrm{eff}} \left(1 - \left(\frac{2\pi\Delta}{k_\mathrm{B}T} \right)^{1/2} \mathrm{e}^{-\Delta/(k_\mathrm{B}T)} \right)$

Zero at a flat band!!!

n_p Particle density

$\frac{1}{m_\mathrm{eff}} \propto J \propto \partial_{k_i} \partial_{k_j} \epsilon_\mathbf{k}$ Bandwidth $\qquad i, j = x, y, z$

Formation of flat bands



Superfluid weight in a multiband system

$$D_s = D_{s,\mathrm{conventional}} + D_{s,\mathrm{geometric}}$$

$$\propto \partial_{k_i}\partial_{k_j}\epsilon_{\mathbf{k}}$$

$$i,j = x,y,z$$

Can be nonzero also in a flat band
Present only in a multiband case
Proportional to quantum metric!

Peotta, Törmä, Nature Communications **6**, 8944 (2015)
Liang, Vanhala, Peotta, Siro, Harju, Törmä, Physical Review B **95**, 024515 (2017)
Huhtinen, Herzog-Arbeitman, Chew, Bernevig, Törmä, Physical Review B **106**, 014518 (2022)

Quantum geometric tensor

Metric for the distance between quantum states

Provost, Vallee, Comm. Math. Phys. **76**, 289 (1980)

$$d\ell^2 = ||u(\mathbf{k}+d\mathbf{k}) - u(\mathbf{k})||^2 = \langle u(\mathbf{k}+d\mathbf{k}) - u(\mathbf{k})|u(\mathbf{k}+d\mathbf{k}) - u(\mathbf{k})\rangle$$

$$\approx \sum_{i,j} \underbrace{\langle \partial_{k_i} u|\partial_{k_j} u\rangle}\, dk_i dk_j$$

Introduce gauge invariant version $\left(u(\mathbf{k}) \leftrightarrow u(\mathbf{k})e^{i\phi(\mathbf{k})}\right)$

➡ Quantum geometric tensor

$$\mathcal{B}_{ij}(\mathbf{k}) = 2\langle \partial_{k_i} u|(1 - |u\rangle\langle u|)|\partial_{k_j} u\rangle$$

$\mathrm{Re}\,\mathcal{B}_{ij} = g_{ij}$ quantum metric $d\ell^2 = \sum_{ij} g_{ij} dk_i dk_j$

$\mathrm{Im}\,\mathcal{B}_{ij} = [\mathbf{\Omega}_{\mathrm{Berry}}]_{ij}$ Berry curvature

Quantum metric is the same as Fubini-Study metric

Chern number: $C = \dfrac{1}{2\pi}\displaystyle\int_{\mathrm{B.Z.}} \mathrm{d}^2\mathbf{k}\, \Omega_{\mathrm{Berry}}(\mathbf{k})$

Lower bound for flat band superfluidity

Peotta, Törmä, Nature Communications **6**, 8944 (2015)

The quantum geometric tensor $\mathcal{B}_{ij}$ is complex positive semidefinite

$$\rightarrow \boxed{D_s \geqslant \int_{B.Z.} d^d\mathbf{k} |\mathbf{\Omega}_{\mathrm{Berry}}(\mathbf{k})| \geqslant C}$$

Time reversal symmetry assumed; C is a spin Chern number

Constituents: interactions, density of states (DOS) and Bloch functions = quantum geometry and topology

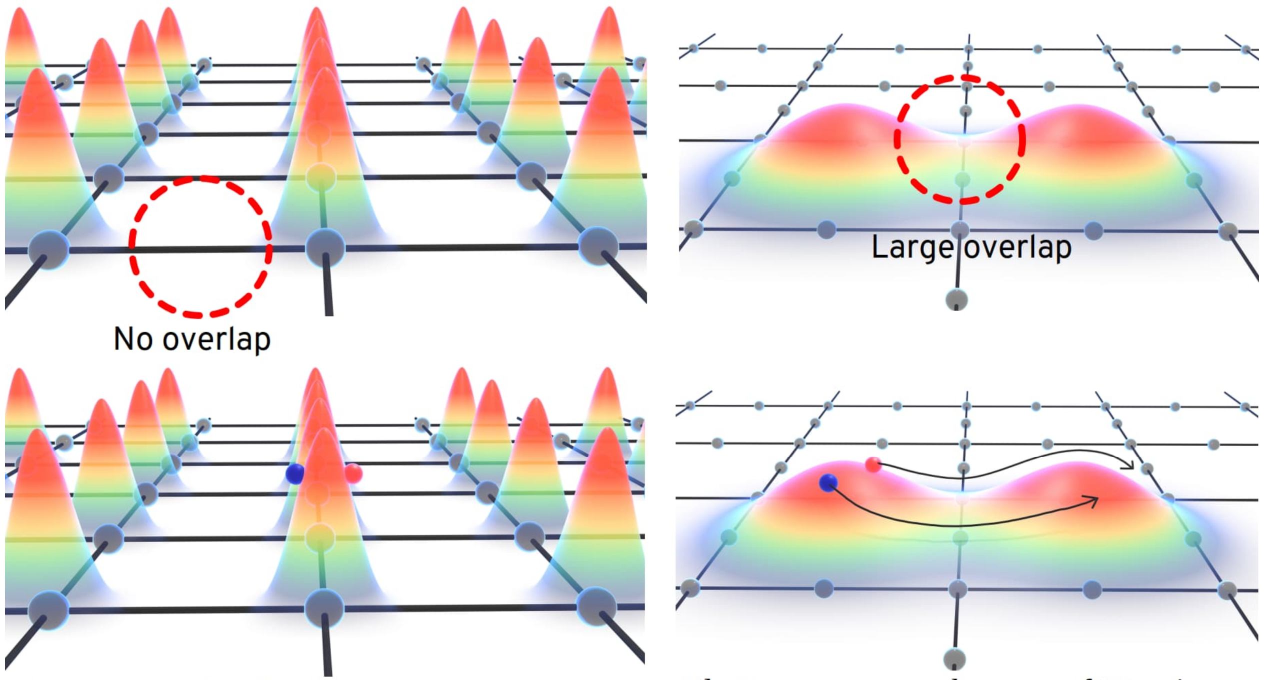
Why can there be transport in a flat band?

Localization and flat band due to vanishing overlap

Localization and flat band due to interference

Non-interacting

Interacting



Electrons stay localized

Electrons can move because of Wannier function overlap

$$C \neq 0 \Leftrightarrow \text{non-localized } w(\mathbf{r}) = \mathcal{F}[u(\mathbf{k})]$$

Brouder, Panati, Calandra, Marzari, Physical Review Letters **98**, 046402 (2007)

$$D_s \propto g_{ij} \geqslant C$$

Twisted Bilayer Graphene (TBG) superconductivity since 2018; first example of flat band superconductivity

Reviews: Balents, Dean, Efetov, Young, Nature Physics 2020
Andrei, Efetov, Jarillo-Herrero, MacDonald, Mak, Senthil, Tutuc, Yazdani, Young, Nature Reviews Materials 2021

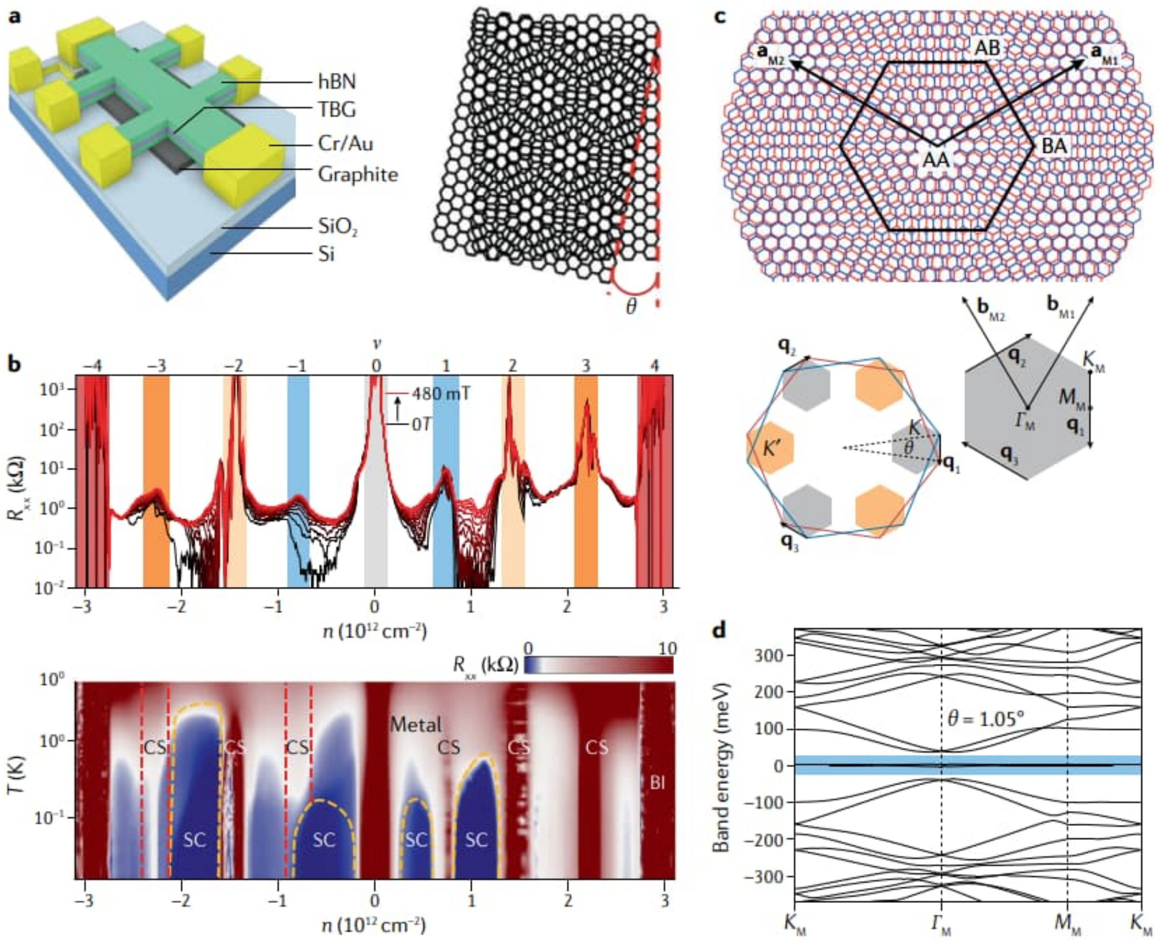


Figure credits see Fig.1 in Törmä, Peotta, Bernevig, Nature Reviews Physics **4**, 528 (2022)