1A Basic rules of probability

In this first exercise there are four class problems, and two home problems. It is probably useful to consult the Ross textbook (see link on course page) and/or the lecture slides (lecture 1A).

Class problems

1A1 (Rolling a die twice) A regular six-sided die is rolled twice. The set of all possible results (the *sample space*) is the following:

$$\begin{aligned} S &= \{(x,y) : x = 1,2,3,4,5,6 \ \text{ and } \ y = 1,2,3,4,5,6\} \\ &= \{(1,1),(1,2),(1,3),\ldots,(6,6)\}, \end{aligned}$$

where x = result from the first roll, and y = result from the second roll. Thus the sample space has 36 elements, each of which is a pair of integers. For example, the element $(2,3)$ means that we first rolled a two, and then a three. Now *draw* the sample space as a grid of 6×6 cells. For each of the following events, *color* the cells corresponding to the event, and *determine* the probability of the event.

(a) $A = \{(x,y) \in S : x = 1\}$,

(b) $B = \{(x,y) \in S : y \geq 4\}$,

(c) $C = \{(x,y) \in S : x + y = 7\}$,

(d) $A \cup B$ = union of A and B,

(e) $B \cap C$ = intersection of B and C,

(f) B^c = complement of B.

To determine the probability of an event, count its cells, observe that each cell has probability 1/36, and add them up. Recall what union, difference, and complement mean in the context of sets. (Ross section 3.3, or lecture slides 1A.)

1A2 (Soil samples) We analyze a soil sample from a waste dump. With probability 0.40 we find arsenic. With probability 0.30 we find lead. With probability 0.10 we find both.

Note that "finding arsenic" does not mean "finding arsenic only". If we find arsenic, we may or may not find *also* lead.

(a) What is the probability that we find lead but not arsenic?

(b) What is the probability that we find arsenic but not lead?

(c) What is the probability that we find at least one of them?

(d) What is the probability that we find neither?

1A3 (Blue cab) In a certain town there are 100 taxicabs: one is blue, and 99 are green. One night, a cab collides with a bicycle, and flees the scene. An eye witness says that the colliding cab was *blue*. However, from previous research we know that in similar situations, a blue car is seen as blue with probability 90%; and a *green* car is seen as blue with probability 8%.

What is the probability that the colliding cab was indeed blue? Do you think that the driver of the only blue cab in town should be convicted, based on this evidence alone?

1A4 (Sampling) A small village has 120 inhabitants, and 20 of them landowners. For convenience we label the inhabitants with integers $1, 2, \ldots, 120$ so that $1, 2, \ldots, 20$ are the landowners.

(a) One inhabitant is picked at random. What is the probability that (s)he is a landowner? Call this number p.

(b) **Sampling with replacement.** Three times we do this: Pick an inhabitant at random from the 120 inhabitants. (We can pick the same inhabitant again.) Let the inhabitants, in the order we picked them, be (i, j, k). How many different choices are there? Call this number N.

(c) In the previous item's scenario, how many different choices there are where all three inhabitants are landowners? Call this number A.

(d) Calculate A/N to five decimals. This is the probability that we pick three landowners. Compare to p^3 and explain. Hint: Think of rolling a die.

(e) **Sampling without replacement.** We pick three inhabitants (i, j, k) thus: The first is chosen at random from all inhabitants. The second is chosen from the remaining ones, and then the third from the remaining ones. How many different choices are there? Call this number M.

(f) In the scenario of the previous item, how many different choices are there where all three inhabitants are landowners? Call this number B.

(g) Calculate B/M to five decimals. This is the probability that we pick three landowners. Compare to previous results and explain.

Hint: Ordered sequences (lecture 1A), principle of counting (Ross §3.5).

Home problems

1A5 (Slot machines) At a casino there are two kinds of slot machines, which look quite identical from the outside. In machines of type A, the winning probability is 5%. In machines of type B, the winning probability is 7%. We also know that 70% of the machines are of type A, and 30% are of type B.

(a) A gambler picks a slot machine at random, and plays once. What is the probability that he wins?

(b) If the gambler wins, what is the probability that the machine is of type B?

1A6 (Strings of letters) On a certain planet, there are 4^{10} inhabitants, each of which has a unique 10-letter identifier. Each letter is either A, B, C or D, so there are 4^{10} different identifiers, and every identifier is currently in use. Identifiers look like CBAADAABBB or AADABCAACC.

(a) How many of the identifiers are palindromes, that is, the same when read left-to-right and right-to-left? (AABCAACBAA is an example of a palindrome.)

(b) How many identifiers are such that no two consecutive letters are identical? (For example, ABADABACAD is like this, but AACBDBABCB is not.)

An inhabitant of the planet is now chosen uniformly at random (that is, each inhabitant has the same probability of being chosen). What is the probability that the identifier of the chosen inhabitant

(c) is a palindrome?

(d) has no two consecutive identical letters?

It is probably useful to apply the *generalized basic principle of counting* (Ross section 3.5, or lecture slides 1A).