1B Discrete random variables

First familiarize yourself with the notions "random variable" and "distribution", e.g. from lecture 1B slides and from Ross.

It is also useful to recall some basic rules of probability: additivity and the chain rule (multiplication rule).

If an unknown quantity has just a finite number of possible values, we can list the possibilities, and their probabilities of occurring, as a table. We can call the quantity a *random variable*, and the table its *distribution*.

Choosing from some possibilities *equiprobably* means that each possibility has equal probability of being chosen, namely $1/n$, if there are n possibilities.

Class problems

1B1 (Two wolves) On a farm there are four ducks, four geese and two hens. During the night two wolves arrive. Each wolf catches one bird at random. Let us denote

$$\begin{aligned} X &= \text{the number of ducks caught,} \\ Y &= \text{the number of geese caught.} \end{aligned}$$

(a) Determine the distribution of the random variable X, by creating a table of its possible values and their probabilities.

(b) Determine the distribution of Y.

(c) Determine the *joint distribution* of the random variables X and Y by creating a 3×3 table, with cells containing the probabilities of values of the pair (X, Y).

(d) Calculate the row sums and column sums of the table in (c). Compare them to the tables in (a) and (b).

(e) Using the table in (c), calculate the probability for the event that the number of ducks caught equals the number of geese caught.

(f) Using the table in (c), determine whether the random variables X and Y are stochastically dependent or independent.

1B2 (Unknown die.) In a box there are four dice: one 4-sided, one 6-sided, and two 8-sided dice. Each die has its sides numbered as usual, with integers starting from 1. Your friend picks one die from the box randomly and equiprobably, and rolls that die once, without showing it to you. Let us denote

$$\begin{aligned} S &= \text{number of sides on the chosen die,} \\ T &= \text{result of the roll.} \end{aligned}$$

(a) Determine the distribution of S by creating a table with its possible values and their probabilities.

(b) Determine the joint distribution of S and T by creating a 3×8 table of the probabilities for the values of the pair (S, T).

(c) From the table in (b), determine the (marginal) distribution of T.

(d) Your friend tells you that the result of the roll was 3. Determine the *conditional distribution* of S using this information. That is, for each integer i that is a possible value for S, determine the conditional probability $\mathrm{P}(S = i \mid T = 3)$.

Home problems

1B3 (Family planning) A couple decides to have children until at least one of the following conditions is fulfilled:

- They have two girls.
- They have four children in total.

Children are born one at a time, so that each child is a girl or a boy with equal probabilities, independent of the previous children. Let us inspect the situation after the couple has stopped having children (so at least one of the above conditions has been fulfilled). Let the total number of children be $C = G + B$, where

$$\begin{aligned} G &= \text{the number of girls,} \\ B &= \text{the number of boys.} \end{aligned}$$

(a) Determine the distribution of the random variable C. Hint: Consider the possible child sequences in birth order, such as BBG (two boys and then a girl). Think how the sequences are generated, starting from the first child. Determine the probabilities of the sequences, and stop when appropriate.

(b) Determine the joint distribution of G and B.

(c) With what probability does the family have more girls than boys?

(d) Determine the joint distribution of C and G.

1B4 (Extreme values) An ordinary six-sided die is rolled five times. Let the results be X_1, X_2, X_3, X_4, X_5, in that order. Let the greatest result be G. For example, if the results are $1, 4, 3, 2, 2$, then $G = 4$.

(a) What are the probabilities for the events $\{G \leq k\}$, for each of the integers $k = 1, 2, 3, 4, 5, 6$?

It would be possible to list all $6^5 = 7776$ possible result sequences, and collect the cases where, say, $G \leq 3$, but we really do not want to do that by hand. Instead, use this hint. What is the condition that each individual roll has to fulfill so that $G \leq 3$? What is the probability that this condition is fulfilled for all five rolls? Do the same for each k.

(b) Using the results from the previous item, and the additive rule of probability, determine the probabilities for the events $\{G = k\}$, for all $k = 1, 2, 3, 4, 5, 6$. Now present the distribution of G as a table. Explain the rough shape of the distribution in words (or use a picture).

(c) Show that G and X_1 are *dependent* (i.e. not independent), e.g. by showing that equation (4.3.8) [Ross section 4.3.1] is not fulfilled.

Hint: Consider, for example, the events $\{G = 1\}$ and $\{X_1 = 1\}$.