

2A Continuous random variables

Class problems

2A1 (A model for wage distribution.) We decide to model the monthly wage (in euros) earned by a randomly chosen employee with a continuous random variable X that has density

$$f(x) = \begin{cases} \alpha c^\alpha x^{-\alpha-1}, & \text{when } x > c, \\ 0, & \text{otherwise,} \end{cases}$$

with $\alpha = 1.6$ and $c = 1500$.

- What is the cumulative distribution function of X ? Draw a graph of it. **Hint: Integrating a power of x works as you expect even if the exponent is not an integer.**
- What are the possible values of X ? In particular, how small and how large can wages be?
- Calculate the probability that a randomly chosen employee earns more than 15 000 eur.
- Find a real number x such that 90% of employees earn at most x eur.

2A2 (Both late.) In this problem we study a *continuous joint distribution* of two variables in a simple case. From multivariate calculus, we only need the following fact: the integral of a *constant* function over a region equals that constant times the area of the region.

Ursula and Vera have agreed to meet for lunch exactly at noon (12:00). However, Ursula arrives U minutes late, and Vera arrives V minutes late. We assume their arrival times are independent, and both are uniformly distributed over the interval $[0, 60]$.

- Write down the density functions f_U and f_V , and the joint density function $f_{U,V}$.
Start from the individual densities and then deduce the joint density. Read lecture slides 1B and/or Ross's section 4.3.1. Be careful to express where your functional expression is valid. Where is the joint density zero and where is it nonzero?
- Calculate the probability that Ursula arrives before 12:20.
Use the distribution of U only.
- Find geometrically the probability that Ursula arrives before 12:15 and Vera arrives between 12:30 and 12:45.
Color the relevant rectangle in the square $[0, 60] \times [0, 60]$. Calculate the area of this rectangle. Recall the joint density and apply the basic fact from multivariate calculus.
- Calculate the probability in (c) again, without resorting to geometry, from the probabilities of the individual events $\{U < 15\}$ and $\{30 < V < 45\}$. **Exploit the independence of U and V .**
- Find geometrically the probability that Ursula arrives at least 30 minutes after Vera.
You need a bit of reasoning to find which region contains the points corresponding to the event. Try fixing a value of V , and think what must U then be for the event to occur. After you have found and drawn the region, calculate its area. Recall the area of a triangle.

Home problems

2A3 (Device lifetime) A satellite orbiting the Earth contains a device whose lifetime X (in years) has the *exponential distribution* with parameter $\lambda = 0.5$. Find its density function from lecture slides (or look up Ross's section 5.6).

- Show how the cumulative distribution function $F(t)$ is obtained by integrating the density.
- Directly from the CDF, calculate the probabilities of the events $A = \{X \leq 1\}$, $B = \{X > 5\}$, and $C = \{5 < X \leq 6\}$. Give the results with at least 6 decimals. Explain in words what these events are.
- From the numerical results in (b), and using the definition of conditional probability, calculate $P(C|B)$. Compare to $P(A)$ and explain.
- Consider a very short interval of $h = 0.01$ years. If the device has lasted to a certain point in time, what is the probability that it breaks during the *next* 0.01 years? Compare your numerical result to the value of λh , and explain why λ is called the *rate parameter*.

2A4 (Fuzzy logic.) In fuzzy logic, a proposition (a claim that something is true) has, instead of a binary truth value (0 or 1), a real-valued truth value in the interval $[0, 1]$. Researchers are modelling a particular proposition's truth value with the random variable X that has density

$$f(x) = \begin{cases} cx(1-x), & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Determine the constant c . **The density must integrate to 1.**
- Determine the cumulative distribution function of X , and draw it.
- Calculate the probability that the truth value is at least 0.75.
- Find the *mode* of the distribution, that is, the point x where the density $f(x)$ attains its maximum.