

MEC-E8001 Finite Element Analysis; Formulae

GENERAL

Displacement: $\vec{u} = u_X \vec{I} + u_Y \vec{J} + u_Z \vec{K} = u_x \vec{i} + u_y \vec{j} + u_z \vec{k} = u\vec{i} + v\vec{j} + w\vec{k}$

Rotation (small): $\vec{\theta} = \theta_X \vec{I} + \theta_Y \vec{J} + \theta_Z \vec{K} = \theta_x \vec{i} + \theta_y \vec{j} + \theta_z \vec{k} = \phi\vec{i} + \theta\vec{j} + \psi\vec{k}$

Coordinate systems:
$$\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}$$

Stress-strain:
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = [E] \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Elasticity matrices: $[C] \equiv [E], [C]_{\sigma} \equiv [E]_{\sigma}, G = \frac{E}{2(1+\nu)}$

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}, \quad [E]^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad [E]_{\epsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Strain-displacement:
$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix}$$

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u / \partial x)^2 + (\partial v / \partial x)^2 + (\partial w / \partial x)^2 \\ (\partial u / \partial y)^2 + (\partial v / \partial y)^2 + (\partial w / \partial y)^2 \\ (\partial u / \partial z)^2 + (\partial v / \partial z)^2 + (\partial w / \partial z)^2 \end{Bmatrix}$$

$$\begin{Bmatrix} E_{xy} \\ E_{yz} \\ E_{zx} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u / \partial x)(\partial u / \partial y) + (\partial v / \partial x)(\partial v / \partial y) + (\partial w / \partial x)(\partial w / \partial y) \\ (\partial u / \partial y)(\partial u / \partial z) + (\partial v / \partial y)(\partial v / \partial z) + (\partial w / \partial y)(\partial w / \partial z) \\ (\partial u / \partial z)(\partial u / \partial x) + (\partial v / \partial z)(\partial v / \partial x) + (\partial w / \partial z)(\partial w / \partial x) \end{Bmatrix}$$

PRINCIPLE OF VIRTUAL WORK

$$\delta W = \sum_{e \in E} \delta W^e = 0 \quad \forall \delta \mathbf{a}, \quad \delta W = \int_{\Omega} \delta w d\Omega$$

$$\mathbf{Bar} \text{ (x): } \delta w_{\Omega}^{\text{int}} = -\frac{\partial \delta u}{\partial x} EA \frac{\partial u}{\partial x}, \quad \delta w_{\Omega}^{\text{ext}} = \delta u f_x, \quad \delta w_{\Omega}^{\text{ine}} = -\delta u \rho A \frac{\partial^2 u}{\partial t^2}, \quad \delta w_{\Omega}^{\text{cpl}} = \frac{d\delta u}{dx} EA \alpha \Delta \vartheta$$

$$\delta w_{\Omega^{\circ}}^{\text{int}} = -\delta E_{xx} A^{\circ} C E_{xx}, \quad \delta w_{\Omega^{\circ}}^{\text{ext}} = A^{\circ} \rho^{\circ} (\delta u g_x + \delta v g_y + \delta w g_z)$$

$$\delta p_{\Omega}^{\text{int}} = -\frac{d\delta \vartheta}{dx} kA \frac{d\vartheta}{dx}, \quad \delta p_{\Omega}^{\text{ext}} = \delta \vartheta s$$

$$\mathbf{Torsion} \text{ (x): } \delta w_{\Omega}^{\text{int}} = -\frac{\partial \delta \phi}{\partial x} GJ \frac{\partial \phi}{\partial x}, \quad \delta w_{\Omega}^{\text{ext}} = \delta \phi m_x, \quad \delta w_{\Omega}^{\text{ine}} = -\delta \phi \rho J \frac{\partial^2 \phi}{\partial t^2}$$

$$\mathbf{Bending} \text{ (xz): } \delta w_{\Omega}^{\text{int}} = -\frac{\partial^2 \delta w}{\partial x^2} EI_{yy} \frac{\partial^2 w}{\partial x^2}, \quad \delta w_{\Omega}^{\text{ext}} = \delta w f_z$$

$$\delta w_{\Omega}^{\text{ine}} = -\delta w \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial \delta w}{\partial x} \rho I_{yy} \frac{\partial^2 \partial w}{\partial t^2 \partial x}, \quad \delta w_{\Omega}^{\text{sta}} = -\frac{d\delta w}{dx} N \frac{dw}{dx} \quad \text{where } N = EA \frac{du}{dx}.$$

$$\mathbf{Bending} \text{ (xy): } \delta w_{\Omega}^{\text{int}} = -\frac{\partial^2 \delta v}{\partial x^2} EI_{zz} \frac{\partial^2 v}{\partial x^2}, \quad \delta w_{\Omega}^{\text{ext}} = \delta v f_y,$$

$$\delta w_{\Omega}^{\text{ine}} = -\delta v \rho A \frac{\partial^2 v}{\partial t^2} - \frac{\partial \delta v}{\partial x} \rho I_{zz} \frac{\partial^2 \partial v}{\partial t^2 \partial x}, \quad \delta w_{\Omega}^{\text{sta}} = -\frac{d\delta v}{dx} N \frac{dv}{dx} \quad \text{where } N = EA \frac{du}{dx}$$

Thin-slab (xy):

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta u / \partial y + \partial \delta v / \partial x \end{Bmatrix}^T t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T t\rho \frac{\partial^2}{\partial t^2} \begin{Bmatrix} u \\ v \end{Bmatrix},$$

$$\delta w_{\Omega^{\circ}}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^T t[C]_{\sigma} \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix}, \quad \delta w_{\Omega^{\circ}}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \rho^{\circ} t^{\circ} \begin{Bmatrix} g_x \\ g_y \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial x \end{Bmatrix}^T \frac{E\alpha}{1-\nu} \int \Delta \vartheta dz \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial \delta \vartheta / \partial x \\ \partial \delta \vartheta / \partial y \end{Bmatrix}^T tk \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{ext}} = \delta \vartheta s$$

Bending (xy):

$$\delta w_{\Omega}^{\text{int}} = - \begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^T \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \delta w f_z$$

$$\delta w_{\Omega}^{\text{ine}} = - \begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^T \frac{t^3}{12} \rho \frac{\partial^2}{\partial t^2} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} - \delta w t \rho \frac{\partial^2 w}{\partial t^2}$$

$$\delta w_{\Omega}^{\text{sta}} = - \begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^T \begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t [E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{cpl}} = - \begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \end{Bmatrix}^T \frac{\alpha E}{1-\nu} \int z \Delta \vartheta dz \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Solid:

$$\delta w_{\Omega}^{\text{int}} = - \begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta w / \partial z \end{Bmatrix}^T [E] \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix} - \begin{Bmatrix} \partial \delta u / \partial y + \partial \delta v / \partial x \\ \partial \delta v / \partial z + \partial \delta w / \partial y \\ \partial \delta w / \partial x + \partial \delta u / \partial z \end{Bmatrix}^T G \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{int}^{\circ}} = - \begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ \delta E_{zz} \end{Bmatrix}^T [C] \begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta E_{xy} \\ \delta E_{yz} \\ \delta E_{zx} \end{Bmatrix}^T 4G \begin{Bmatrix} E_{xy} \\ E_{yz} \\ E_{zx} \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta w / \partial z \end{Bmatrix}^T \frac{E\alpha}{1-2\nu} \Delta \vartheta \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{int}} = - \begin{Bmatrix} \partial \delta \vartheta / \partial x \\ \partial \delta \vartheta / \partial y \\ \partial \delta \vartheta / \partial z \end{Bmatrix}^T k \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \\ \partial \vartheta / \partial z \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{ext}} = \delta \vartheta s$$

APPROXIMATIONS (some) $u = \mathbf{N}^T \mathbf{a}$, $\xi = \frac{x}{h}$

$$\text{Quadratic: } \mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} \quad (\text{bar})$$

$$\mathbf{Cubic: N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} u_{10} \\ u_{11} \\ u_{20} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ u_{z2} \\ -\theta_{y2} \end{Bmatrix} \text{ beam } xz\text{-plane bending)}$$

$$\mathbf{Linear: N} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

$$\mathbf{VIRTUAL WORK EXPRESSIONS} \quad \ddot{a} \equiv \frac{d^2}{dt^2} a$$

$$\mathbf{Rigid body/point force: } \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^T \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_{X1} \\ \delta \theta_{Y1} \\ \delta \theta_{Z1} \end{Bmatrix}^T \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}$$

$$\delta W^{\text{ine}} = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{y1} \\ \delta u_{z1} \end{Bmatrix}^T m \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{y1} \\ \ddot{u}_{z1} \end{Bmatrix} - \begin{Bmatrix} \delta \theta_{x1} \\ \delta \theta_{y1} \\ \delta \theta_{z1} \end{Bmatrix}^T \begin{Bmatrix} J_{xx} \ddot{\theta}_{x1} \\ J_{yy} \ddot{\theta}_{y1} \\ J_{zz} \ddot{\theta}_{z1} \end{Bmatrix}$$

$$\mathbf{Bar: } \delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\delta W^{\text{ine}} = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \frac{\rho Ah}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}, \quad \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \Delta \mathcal{G}_1 \\ \Delta \mathcal{G}_2 \end{Bmatrix}$$

$$\delta P^{\text{int}} = - \begin{Bmatrix} \delta \mathcal{G}_1 \\ \delta \mathcal{G}_2 \end{Bmatrix}^T \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{Bmatrix}, \quad \delta P^{\text{ext}} = \begin{Bmatrix} \delta \mathcal{G}_1 \\ \delta \mathcal{G}_2 \end{Bmatrix}^T \frac{sh}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\delta W^{\text{int}} = -\delta h \frac{h}{h^\circ} CA^\circ \frac{1}{2} \left[\left(\frac{h}{h^\circ} \right)^2 - 1 \right], \quad \delta W^{\text{ext}} = \begin{Bmatrix} g_x \delta u_{x1} + g_y \delta u_{y1} + g_z \delta u_{z1} \\ g_x \delta u_{x2} + g_y \delta u_{y2} + g_z \delta u_{z2} \end{Bmatrix}^T \frac{\rho^\circ A^\circ h^\circ}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$h^2 = (h^\circ + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2$$

$$\mathbf{Torsion: } \delta W^{\text{int}} = - \begin{Bmatrix} \delta \theta_{x1} \\ \delta \theta_{x2} \end{Bmatrix}^T \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta \theta_{x1} \\ \delta \theta_{x2} \end{Bmatrix}^T \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\delta W^{\text{ine}} = - \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^T \frac{\rho J h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{x1} \\ \ddot{\theta}_{x2} \end{Bmatrix}$$

Bending (xz):

$$\delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^T \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^T \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

$$\delta W^{\text{ine}} = - \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^T \left(\frac{\rho I_{yy}}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} + \frac{\rho Ah}{420} \begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}$$

$$\delta W^{\text{sta}} = - \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^T \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} \quad \text{where } N = EA \left(\frac{u_{x2} - u_{x1}}{h} \right)$$

Bending (xy):

$$\delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^T \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^T \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix}$$

$$\delta W^{\text{ine}} = - \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^T \left(\frac{\rho I_{zz}}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} + \frac{\rho Ah}{420} \begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{Bmatrix}$$

$$\delta W^{\text{sta}} = - \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^T \frac{N}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix} \quad \text{where } N = EA \left(\frac{u_{x2} - u_{x1}}{h} \right)$$

CONSTRAINTS

Frictionless contact: $\vec{n} \cdot \vec{u}_A = 0$

Joint: $\vec{u}_B = \vec{u}_A$

Rigid body (link): $\vec{u}_B = \vec{u}_A + \vec{\theta}_A \times \vec{\rho}_{AB}$, $\vec{\theta}_B = \vec{\theta}_A$.

MATHEMATICS

Polar representation: $e^{i\alpha} = \cos \alpha + i \sin \alpha$, $\sin i\alpha = i \sinh \alpha$, $\cos i\alpha = \cosh \alpha$, $i^2 = -1$

Eigenvalue decomposition: $\mathbf{A} = \mathbf{X}\lambda\mathbf{X}^{-1}$

Matrix function: If $\mathbf{A} = \mathbf{X}\lambda\mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X}f(\lambda)\mathbf{X}^{-1}$

Newton's method: If $\mathbf{a} = \mathbf{a} - \left(\frac{\partial \mathbf{R}(\mathbf{a})}{\partial \mathbf{a}}\right)^{-1} \mathbf{R}(\mathbf{a}) \equiv \mathbf{G}(\mathbf{a})$, then $\mathbf{R}(\mathbf{a}) = 0$

Taylor series: $f(x+a) = \sum_{i=0}^n \frac{1}{i!} \left(a \frac{d}{dx}\right)^i f(x) + \frac{1}{(n+1)!} f^{(n+1)}(\underline{x}) a^{n+1}$ $\underline{x} \in [x, x+a]$

TIME INTEGRATION (free vibrations)

Crank-Nicholson: $\begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \begin{bmatrix} \mathbf{I} & -\mathbf{I}/2 \\ \boldsymbol{\alpha}/2 & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{I}/2 \\ -\boldsymbol{\alpha}/2 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)}$, $\boldsymbol{\alpha} = \mathbf{M}^{-1} \mathbf{K} \Delta t^2$

Disc. Galerkin: $\begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{I} - \boldsymbol{\alpha}/2 \\ -\mathbf{I} - \boldsymbol{\alpha}/2 & \boldsymbol{\alpha}/3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)}$, $\boldsymbol{\alpha} = \mathbf{M}^{-1} \mathbf{K} \Delta t^2$