Abstract Algebra – Additional problems

Milo Orlich

The problems listed here are not discussed during the exercise sessions.

Binary operations, groups and subgroups

Exercise 1. For each of the binary operations defined below, decide which of the following properties are satisfied: associativity, commutativity, having a neutral element.

1. On an arbitrary set X, we define $x * y := y$ for all $x, y \in X$.
2. On an arbitrary set X, we fix $a \in X$ and we define $x * y := a$ for all $x, y \in X$.
3. On $X = \mathbb{R}$, we define $x * y := \min(x, y)$.
4. On $X = \mathbb{R}$, we define $x * y := |x + y|$.
5. On $X = \mathbb{N}$, we define $x * y := xy + 1$.
6. On $X = \mathbb{N}$, we define $x * y := x^2 y$.
7. On $X = \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\}$, we define $f * g := (fg)'$ (the derivative of the product).
8. For an arbitrary set X, on the powerset $P(X)$, we define $A * B := (A \cup B) \setminus (A \cap B)$ for all $A, B \in P(X)$.

Exercise 2. Let $a, b, c, d \in \mathbb{Q}$. Define a binary operation $*$ on $\mathbb{Q}$ by setting

$$x * y := axy + bx + cy + d.$$

Determine all the values of a, b, c, d for which $*$

- is associative
- is commutative
- has a neutral element

Deduce that "in almost all cases", if $*$ is associative, then it is also commutative and has a neutral element. *Hint:* If $f \in \mathbb{Q}[x, y]$ is a nonzero polynomial, then there exist $a, b \in \mathbb{Q}$ such that $f(a, b) \neq 0$. (This holds for a nonzero polynomial with coefficients in an infinite field in an arbitrary number n of variables, and it can be shown by induction on n. It does not hold for finite fields.)

Exercise 3. Let X be a set and $f\colon X \to X$ a function. Define $x * y = f(y)$ for all $x, y \in X$. What condition does f have to satisfy in order for $*$ to be associative? For $X = \{1,2,3,4\}$, determine f so that the associativity condition $(x*y)*z = x*(y*z)$ holds for no triple (x,y,z) of elements of X.

Exercise 4. Compute the order of the followig elements:

1. The complex numbers 2, $-i$, $1+i$, $\cos(\pi/7) + i\sin(\pi/7)$ with respect to the product.
2. All elements of the group $\mathcal{U}(Z_8)$, with respect to the product modulo 8.
3. All elements of S_4.

Exercise 5. Consider the set with a single element a. Which binary operation turns this set into a group? Check the group axioms. Do the same for a set with exactly two elements a and b.

Exercise 6. Which of the following pairs is a group?

$$(\mathbb{N},+), \quad (\mathbb{N},\cdot), \quad (\mathbb{N}\setminus\{0\},\cdot), \quad (\mathbb{Z},+), \quad (\mathbb{Z},\cdot), \quad (\mathbb{Z}\setminus\{0\},\cdot), \quad (\mathbb{R}_{\geq 0},+), \quad (\mathbb{R}_{>0},\cdot),$$

$$(\mathbb{R}^3,+), \qquad (\mathbb{N},*) \text{ with } m*n := m^n, \qquad (\mathrm{GL}_n(\mathbb{Q}),\cdot), \qquad (\mathbb{R},\min),$$

$$(\mathbb{Z}_7\setminus\{\overline{0}\},\cdot), \qquad (\mathbb{Z}_9\setminus\{\overline{0}\},\cdot), \qquad (\{f\colon \mathbb{N}\to\mathbb{N} \mid f \text{ is injective}\},\circ).$$

In each case, if it is a group, is it abelian?

Exercise 7. Consider the group $(\mathbb{C},+)$. Which of the following are subgroups?

$$\{z\in\mathbb{C} \mid |z|=1\} \qquad \{z\in\mathbb{C} \mid \mathrm{Re}(z)=\mathrm{Im}(z)\} \qquad \{z\in\mathbb{C} \mid \mathrm{Im}(z)\geq 0\}$$
$$\{z\in\mathbb{C} \mid |z|\leq 1\} \qquad \{z\in\mathbb{C} \mid \mathrm{Re}(z)\in\mathbb{Z}\} \qquad \{z\in\mathbb{C} \mid \mathrm{Im}(z)\geq 1\}$$

Exercise 8. Consider the group $(\mathbb{C}\setminus\{0\},\cdot)$. Which of the following are subgroups?

$$\{z\in\mathbb{C}\setminus\{0\} \mid |z|=1\} \qquad \{z\in\mathbb{C}\setminus\{0\} \mid \mathrm{Re}(z)=\mathrm{Im}(z)\} \qquad \{z\in\mathbb{C}\setminus\{0\} \mid \mathrm{Im}(z)\geq 0\}$$
$$\{z\in\mathbb{C}\setminus\{0\} \mid |z|\leq 1\} \qquad \{z\in\mathbb{C}\setminus\{0\} \mid \mathrm{Re}(z)\in\mathbb{Z}\} \qquad \{z\in\mathbb{C}\setminus\{0\} \mid \mathrm{Im}(z)\geq 1\}$$

Exercise 9. Consider the group $(\mathrm{GL}_2(\mathbb{Q}),\cdot)$. Which of the following are subgroups?

$$\{A\in\mathrm{GL}_2(\mathbb{Q}) \mid \det(A)=1\} \qquad \{A=(a_{ij}\in\mathrm{GL}_2(\mathbb{Q}) \mid a_{ij}\in\mathbb{Z}\} \qquad \{A\in\mathrm{GL}_2(\mathbb{Q}) \mid A^2=I\}$$

$$\{A=(a_{ij}\in\mathrm{GL}_2(\mathbb{Q}) \mid a_{ij}\in\mathbb{Z}, \det(A)=\pm 1\} \qquad \{\{A\in\mathrm{GL}_2(\mathbb{Q}) \mid A \text{ is upper triangular}\}$$

$$\{\{A\in\mathrm{GL}_2(\mathbb{Q}) \mid A \text{ is upper triangular, with 1's on the diagonal}\}$$

Among these, are there any abelian subgroups?

Exercise 10. Let $G = S_3$. Determine all subgroups of G, and for each of them say whether it is abelian (and if yes, whether it is cyclic). Verify the formula $a^{\#G} = \mathrm{id}_{\{1,2,3\}}$ for all $a\in G$.

Exercise 11. Let $G = S_4$.

1. Let $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. Determine the order of τ and $\#H$.
2. Let $H := \{\sigma\in G \mid \sigma(4)=4\}$. Show that H is a subgroup of G (isomorphic to what?). Determine $\#H$. Is H abelian?

3. Let $H := \{\sigma \in G \mid \sigma(\{1,2\}) = \{1,2\}\}$. Show that H is a subgroup of G (isomorphic to what?). Determine $\#H$. Is H abelian? Is H cyclic?

Exercise 12. Show that in a non-abelian group with 8 elements there is always an element of order 4. (Use something from Problem set 1.)

Exercise 13. Find all elements of finite order in the following groups:

$$(\mathbb{Z},+), \qquad (\mathbb{Q},+), \qquad (\mathbb{R},+), \qquad (\mathbb{C},+), \qquad (\mathbb{Q}\setminus\{0\},\cdot), \qquad (\mathbb{R}\setminus\{0\},\cdot), \qquad (\mathbb{C}\setminus\{0\},\cdot),$$

$$(\{A \in \mathrm{GL}_n(\mathbb{R}) \mid \{A \text{ is upper triangular}\},\cdot).$$

Exercise 14. Consider the group $G := \mathbb{Z}_2^n$, that is, the product of n copies of $\mathbb{Z}_2$. Determine $\#G$ and the order of all elements of G. How many cyclic subgroups does G have?

Exercise 15. For all $n = 3,\ldots,12$, denote by $U(\mathbb{Z}_n)$ the group consisting of the elements of $\mathbb{Z}_n$ that are invertible with respect to the product (equipped with the product). For each $n = 3,\ldots,12$, check whether $U(\mathbb{Z}_n)$ is cyclic.

Exercise 16. Let G be a group and define

$$Z(G) := \{g \in G \mid gx = xg \text{ for all } x \in G\}$$

the *center* of G.

1. Show that $Z(G)$ is an abelian subgroup of G.
2. Show that $Z(G) = G$ if and only if G is abelian.
3. Show that $Z(S_3) = \{\mathrm{id}\}$.
4. Show that $Z(\mathrm{GL}_2(\mathbb{C}))$ consists of the nonzero multiples of the identity matrix.

Homomorphisms

Exercise 17. Determine which of the following are group homomorphisms:

$$\begin{aligned} \mathbb{Z}^2 &\longrightarrow \mathbb{Z} \\ (x,y) &\longmapsto x+y \end{aligned} \qquad\qquad \begin{aligned} \mathbb{Z}^2 &\longrightarrow \mathbb{Z} \\ (x,y) &\longmapsto xy. \end{aligned}$$

Exercise 18. Let G be a group. Determine which of the following are group homomorphisms:

1. The map $f\colon G \to G$ defined by $f(x) := x^{-1}$.
2. The map $f\colon G \to G$ defined by $f(x) := x^3$.
3. Fixed an element $g \in G$, the map $f\colon G \to G$ defined by $f(x) := xg$.
4. Fixed an element $g \in G$, the map $f\colon G \to G$ defined by $f(x) := gxg$.
5. Fixed an element $g \in G$, the map $f\colon G \to G$ defined by $f(x) := gxg^{-1}$.
6. The map $f\colon G^2 \to G$ defined by $f(x,y) := xy^{-1}$.

Exercise 19. Show that $(\mathbb{R},+)$ is isomorphic to $(\mathbb{R}_{>0},\cdot)$.

Exercise 20. Show that every homomorphism from $(\mathbb{Q}, +)$ to $(\mathbb{Q} \setminus \{0\}, \cdot)$ is constantly equal to 1. Deduce that $(\mathbb{Q}, +)$ is not isomorphic to any subgroup of $(\mathbb{Q} \setminus \{0\}, \cdot)$.

Exercise 21. Let $G = \mathrm{GL}_n(K)$, where K is $\mathbb{R}$ or $\mathbb{C}$. Consider the map $f : G \to G$ defined by $f(A) := \det(A)A$.

1. Show that f is a group homomorphism.
2. Show that $\mathrm{SL}_n(K$ is contained in the image of f.
3. Show that $\ker(f) = \{\lambda I_n \mid \lambda \in K \text{ and } \lambda^{n+1} = 1\}$.
4. If $K = \mathbb{C}$, show that f is surjective.
5. If $K = \mathbb{R}$, show that f is surjective if and only if $n + 1$ is odd.