Suggested Solutions for Final

1. (a) Consider sharing two indivisible items between two agents. Let $x \in \{0,1,2\}$ denote the number of items to 1 and therefore 2 gets $2-x$. Let $u_1(x) = x^2$ and $u_2(x) = (2-x)^2$. Then all $x \in \{0,1,2\}$ are Pareto-efficient in a single allocation problem. In the two-period problem, $(1,1)$ is Pareto-dominated by $(2,0)$ and $(0,2)$, but $(0,0)$ and $(2,2)$ are Pareto efficient.

   (b) A competitive equilibrium is an allocation and a price vector such that all agents choose their most preferred house in their budget set (price less than or equal to the price of initial endowment) and markets clear. To find eq. allocation: top trading cycle: $(2,3,4)$ resulting in allocation (d,a,c,b). Any price vector with $p_a = p_b = p_c > p_d$ supports this

2. (a) First Welfare Theorem: In an exchange economy with locally non-satiated preferences, all CE allocations are PE.

   (b) Interior PE: $MRS_A = MRS_B$ yielding:

   $$\alpha\beta = x_{11}x22.$$

   Resource constraint: $x_{22} = 5 - x_{12}$ so that

   $$x_{12} = 5 - \frac{\alpha\beta}{x_{11}}, x_{21} = 3 - x_{11}, x_{22} = 5 - x_{12}.$$

   Boundary PE: $\{(x_{11},0),(3-x_{11},5)\}$ for $x_{11} \leq \frac{\alpha\beta}{5}$, $\{(3,x_{12}),(0,5-x_{12})\}$ for $x_{12} \geq 5-\frac{\alpha\beta}{3}$.

   (c) From above, initial allocation is PE. Since preferences are convex, second welfare theorem implies that for some prices, the initial allocation is a CE allocation. Interior optimality implies that $\frac{p_1}{p_2} = MRS_A = MRS_B = 1$.

3. (a) Let y_i denote savings of agent i. Then consumption in period 0 for type i is $2 - y_i$. Consumption in both states for type 1 is $1 + y_1$ and for type 2 it is $2 + y_2$ in both states. Optimization:

   $$\max_{y_i} \ln(w_{i0} - y_i) + p_1 \ln(w_{i1} + y_i) + p_2 \ln(w_{i2} + y_i) = \ln(w_{i0} - y_i) + \ln(w_{i1} + y_i).$$

   FOC implies that $w_{i0} - y_i = w_{i1} - y_i$ so that $y_i = \frac{w_{i0}-w_{i1}}{2}$ so that $y_1 = \frac{1}{2}, y_2 = 0$.

   (b) Let q be the price of the asset and z_i demand by i for the asset. Then optimization:

   $$\max_{z_i} \ln(w_{i0} - qz_i) + \ln(w_{i1} + z_i).$$

   Market clearing:

   $$z_1 + z_2 = 0.$$

   Solving gives: $q = \frac{4}{3}, z_1 = -z_2 = \frac{1}{4}$.

   (c) In this case, the maximization is:

   $$\max_{z_i} \ln(w_{i0} - qz_i) + p_i \ln(w_{i1} + z_i).$$

   Otherwise same as before.