General information

The exercise sessions will be held as blackboard sessions, where the participants will present their solutions to the group. As such, the problems should be set up and solved before the session. The focus of the exercises lies on analyzing and discussing the task at hand together with the group: thus, a perfect solution is not required to be awarded points. A point will be awarded for each completed exercise, and a person will be chosen to present their solution from the pool. *There are useful formulas at the bottom of this document!*

Exercise 1.
The fusion reaction

(a) The chemical reaction describing the complete combustion of gasoline is

$$2\ \mathrm{C_8H_{18}} + 25\ \mathrm{O_2} \rightarrow 16\ \mathrm{CO_2} + 18\ \mathrm{H_2O} + 94\ \mathrm{eV},$$

releasing an average energy of 0.9 eV per atom. The D-T fusion reaction

$$\mathrm{D} + \mathrm{T} \rightarrow {}^4\mathrm{He} + \mathrm{n} + 17.6\ \mathrm{MeV}$$

releases, on average, 8.8 MeV per atom, a factor of 10^7 times more than the chemical combustion of gasoline per atom. What are the reasons for this difference, and how do these values compare to the ionization potential for e.g. hydrogen?

(b) Show that the neutron carries 14.05 MeV of the 17.6 MeV produced in D-T fusion. What are the implications for fusion control of this? You may neglect the initial kinetic energy of the D and T ions, and only consider the energy released in the reaction.

Solution 1.

(a) The reason for the large difference in the amount of energy released is due to the fundamentally different forces that hold together molecules and atomic nuclei. The atomic nucleus is kept together by the nuclear force, which is short ranged (order of proton radius) but also extremely strong (strong enough to overcome the Couloumb repulsion between positively charged protons in a nucleus, for example). The interatomic bonds rearranged in chemical reactions, such as combustion, however, are based on the Coulomb force, a much weaker interaction. The ionization energy of a hydrogen atom is 13.6 eV, which is approximately an order of magnitude higher than the energy released per atom in combustion. The ionization energy of hydrogen corresponds to the energy difference between the electron in the ground state (as close as possible to the nucleus) and an electron infinitely far away from the nucleus - typically electrons do not undergo such large potential energy changes in chemical reactions. The ionization energy for hydrogen is especially large, because the electron is very close to the nucleus.

(b) The energies required for fusion reactions are of the order of 10 keV, while the energy released is of the order of $Q = 10$ MeV, allowing us to assume that $E_i + E_f \approx E_f$. We also observe that the energy is kinetic, and distributed between the products. This allows us to ignore the initial momenta, and only consider the final momenta, which has to be equally distributed between the products, and opposite in direction. Based on these assumptions we can write

$$\frac{1}{2}m_{He}v_{He}^2 + \frac{1}{2}m_n v_n^2 \approx Q \tag{1}$$

$$m_{He}v_{He} \approx m_n v_n \tag{2}$$

Equation (1) can be re-arranged to

$$\frac{1}{2}m_{He}v_{He}^2 = \frac{Q}{1 + \frac{m_n v_n^2}{m_{He}v_{He}^2}} = \frac{Q}{1 + \frac{v_n}{v_{He}}} = \frac{Q}{1 + \frac{m_{He}}{m_n}}, \tag{3}$$

where equation (2) has been used for the last two equivalences, since

$$m_{He}v_{He} = m_n v_n \leftrightarrow \frac{v_{He}}{v_n} = \frac{m_n}{m_{He}} \tag{4}$$

Thus the energies for the products are

$$\frac{1}{2}m_{He}v_{He}^2 = \frac{Q}{1 + \frac{m_{He}}{m_n}} \tag{5}$$

$$\frac{1}{2}m_n v_n^2 = \frac{Q}{1 + \frac{m_n}{m_{He}}}. \tag{6}$$

Solving for the masses of the helium nucleus and the neutron, with $Q = 17.6$ MeV yields $K_n \approx 14.05$ MeV. Further, it can be shown that the ratio of the kinetic energies of the products are

$$\frac{K_{He}}{K_n} = \frac{m_n}{m_{He}}. \tag{7}$$

The same analysis can be applied to any fusion reaction with two products.

Exercise 2.
Ignition (Lawson) criterion

(a) **Ideal ignition:** Find the ignition temperature of D-T fuel in a thermonuclear fusion reaction, only considering losses due to Bremsstrahlung. Is this a stable or unstable equilibrium? Assume a 50-50 D-T fuel mix and the low temperature ($T < 25$ keV) approximation for the DT fusion velocity-averaged cross-section

$$\langle \sigma v \rangle_{DT} \approx 3.68 \times 10^{-18} T^{-2/3} \exp\left(-19.94 T^{-1/3}\right) \text{ m}^3 \text{ s}^{-1},$$

for $[T] = \text{keV}$.

(b) **Ignition:** Derive the ignition (Lawson) criterion for $n\tau_E$, where τ_E is the energy confinement time, considering power losses due to Bremsstrahlung radiation and thermal conduction $P_{therm} \approx W/\tau_E$, were W denotes the plasma internal energy. Spectral line radiation losses are not considered for simplicity. Plot the obtained relation, using the same approximation for the velocity-averaged cross-section as above. At what temperature does the minimum occur? Compare this to the reference value given in the lecture slides. If there is a difference, why?

(c) **Ideal operation:** Using the reference values of minimum $n\tau_E$ and the temperature where the minimum occurs, calculate the hydrostatic plasma pressure ($p = 2nk_BT$, T in Kelvin) assuming an ITER-like confinement time of 8 s. How does this pressure compare to the pressure at the center of the sun and the atmospheric pressure?

Solution 2.

(a) **Ideal ignition:** Ideal ignition corresponds to a situation with negligible heat transport losses and negligible external heating. The fusion power density for 50-50 DT fuel is given by:

$$P_f = \frac{1}{4} n_e^2 \langle \sigma v \rangle E_{\text{DT}}. \tag{8}$$

The resulting alpha heating power is

$$P_\alpha = \frac{1}{20} n_e^2 \langle \sigma v \rangle E_{\text{DT}}. \tag{9}$$

In power balance, the alpha heating power is equal to Bremsstrahlung radiation losses:

$$P_\alpha = P_{\text{Br}}, \tag{10}$$

$$\frac{1}{20} n_e^2 \langle \sigma v \rangle E_{\text{DT}} = c_B n_e^2 T_e^{1/2}, \tag{11}$$

Equation 11 is plotted in Fig. 1, assuming $n_e = 10^{20}$ m^{-3}, where it can be observed that the ideal ignition temperature according to these calculations is 4.6 keV.

It is also observed that the equilibrium is **unstable** for small perturbations in temperature. For a small reduction in temperature, the Bremsstrahlung losses become greater than the fusion power, further lowering the plasma temperature. For a small increase in temperature, fusion power increases faster than the Bremsstrahlung losses increase. Therefore, the plasma temperature increases even more.

(b) **Ignition including thermal conduction** The thermal energy content of the plasma is

$$W_p = \frac{3}{2} n_e T_e V + \frac{3}{2} n_i T_i V \approx 3 n_e T V, \tag{12}$$

where quasi-neutrality is assumed: $n_i \approx n_e$. The thermal conductive losses from the plasma can be represented by

$$P_{\text{trans}} \approx \frac{W_p}{\tau_{\text{E}}} \approx \frac{3 n_e T V}{\tau_{\text{E}}}. \tag{13}$$

The power balance equation can now be written:

$$P_\alpha = P_{\text{Br}} + P_{\text{transp}} \tag{14}$$

$$\frac{1}{4} \langle \sigma v \rangle E_\alpha V = c_B n_e^2 T_e^{1/2} V + \frac{3 n_e T V}{\tau_{\text{E}}}. \tag{15}$$

This can be re-arranged to give

$$n_e \tau_{\text{E}} = \frac{12T}{\langle \sigma v \rangle E_\alpha - 4 c_B T_e^{1/2}}. \tag{16}$$

This expression is illustrated in Fig. 2. The solid line uses the fit given in the exercise sheet. The fit should only be used for low energies below $T = 25$ keV. Therefore, discrete points with an accurate cross-section values are also given with red circles. It is observed that the minimum value is of about 2e20 m^{-3}s, and this occurs around $T = 20$ keV.

It is also observed that the ignition criterion approaches asymptotically infinity at the ideal ignition temperature, as expected.

(c) **The ignition pressure assuming ITER-like confinement** The minimum $n_e \tau_{\text{E}} \approx 1.5 \times 10^{20}\,\text{s}\,\text{m}^{-3}$. Assuming an ITER like confinement time of 8 s, therefore, means that the ignition pressure is about 150 000 Pa, or 1.5 times the atmospheric pressure. The pressure at the core of the sun is ca. 265 billion atm.

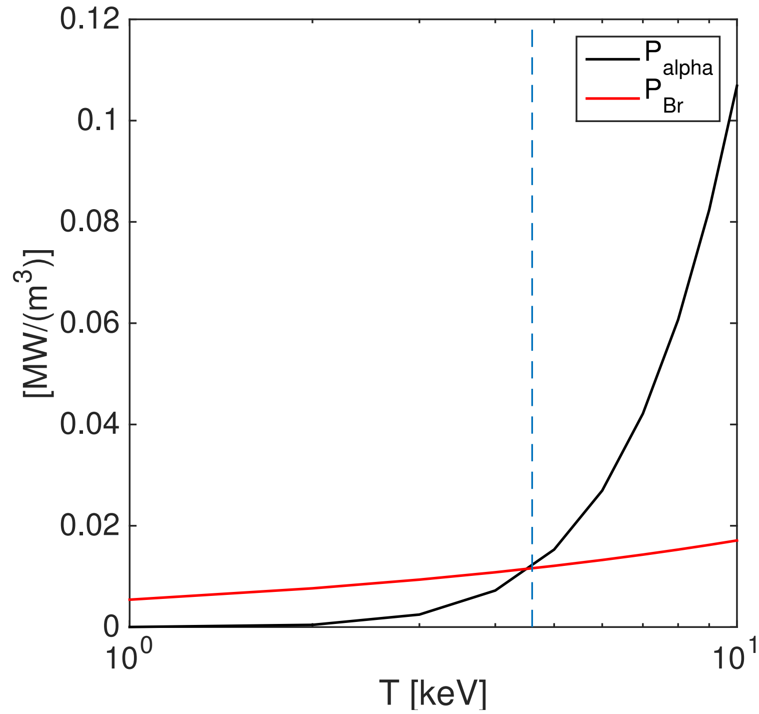


Figure 1: Fusion power density (black) and Bremsstrahlung loss power density (red). The dashed blue line represent the point at which the ideal ignition occurs.

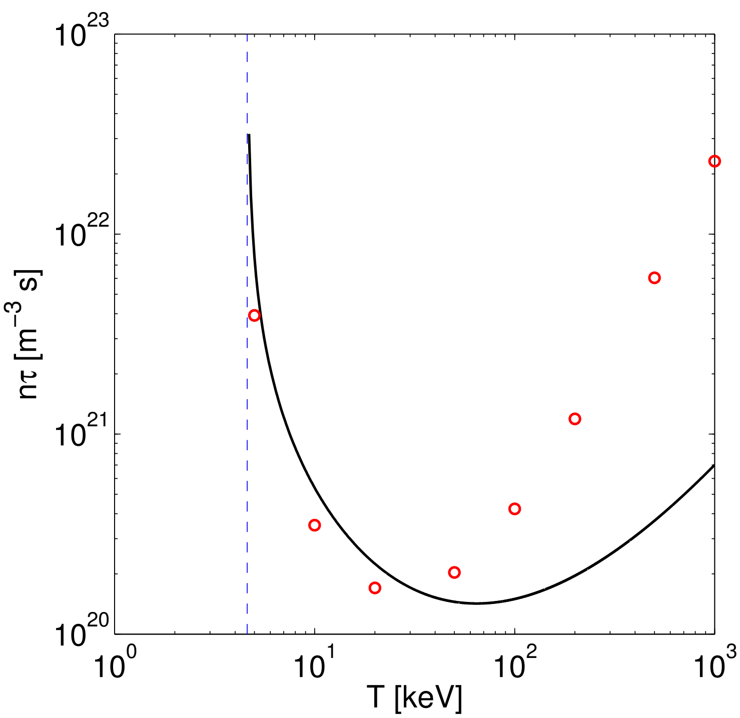


Figure 2: Ignition criterion. The solid black line is drawn using the low temperature fit given in the exercise sheet. The dots represent the accurate cross-section values also given in the exercise sheet. The dashed blue line illustrates the ideal ignition temperature.

Exercise 3.
Plasma power balance and impurity concentration

(a) **Carbon contamination:** Calculate the fusion power reduction due to fuel dilution and impurity radiation density for a $T = 10$ keV, $n_e = 10^{20}$ m^{-3} plasma with 4% $^6C^{6+}$ carbon concentration ($n_C/n_e = 0.04$), and compare the values to each other and to the power density of a pure D-T plasma. The power radiated by carbon at $T = 10$ keV, assuming coronal equilibrium, is $R_C = 10^{-34}$ Wm^3 [1] and the impurity radiation losses are given by

$$P_{rad} = n_Z n_e R_Z \quad [\mathrm{Wm}^{-3}]$$

Set out from the quasi-neutrality condition for plasmas and assume the electron density to have an upper density limit. Assume the carbon to be fully ionized.

(b) **Tungsten contamination:** Perform the same calculations and analysis as above, assuming $n_W/n_e = 10^{-5}$ and that all tungsten atoms are have charge state 50 ($Z = 50$). The power radiated by tungsten at $T = 10$ keV, assuming coronal equilibrium, is $R_W = 10^{-31}$ Wm^3 [1].

(c) What can be said about impurity contamination and fuel dilution based on your findings?

Solution 3.

(a) **Carbon contamination:** Due to quasi-neutrality

$$n_e = n_{\mathrm{DT}} + 6n_{\mathrm{C}^{6+}}, \tag{17}$$

which gives

$$\frac{n_{\mathrm{DT}}}{n_e} = 1 - \frac{6n_{\mathrm{C}^{6+}}}{n_e} = 1 - 6 \cdot 0.04 = 1 - 0.24 = 0.76. \tag{18}$$

In pure plasma conditions

$$n_{\mathrm{DT,\ pure\ plasma}} = n_e. \tag{19}$$

Therefore, the fractional reduction of fuel density due to impurity contamination is

$$\frac{n_{\mathrm{DT}}}{n_{\mathrm{DT,\ pure\ plasma}}} = 0.76. \tag{20}$$

Since the alpha power scales as n_{DT}^2 ($P_\alpha = \frac{1}{20} n_{\mathrm{DT}}^2 \langle \sigma v \rangle E_{\mathrm{DT}}$),

$$\frac{P_\alpha^{\mathrm{diluted}}}{P_\alpha^{\mathrm{pure}}} = (0.76)^2 = 0.58. \tag{21}$$

Therefore, the 4% carbon contamination reduced the alpha heating power by 42 %.

The radiative power loss density can be calculated as

$$P_{\mathrm{rad,\ Z}} = n_{\mathrm{C}^{6+}} n_e R_c = 0.04 \cdot 10^{20} \cdot 10^{20} \cdot 10^{-34} \frac{\mathrm{W}}{\mathrm{m}^3} = 40 \frac{\mathrm{kW}}{\mathrm{m}^3}. \tag{22}$$

The alpha power density in the pure plasma conditions is

$$P_\alpha = \frac{1}{20} n_e^2 \langle \sigma v \rangle E_{\mathrm{DT}} = 1.068 \times 10^5\,\mathrm{W\,m^{-3}} \approx 100\,\mathrm{kW\,m^{-3}} \tag{23}$$

Therefore, the radiative power losses due to carbon are about 40% of the alpha heating power.

(b) **Tungsten contamination:** Repeating the same calculations for tungsten:

$$\frac{n_{\mathrm{DT}}}{n_e} = 1 - \frac{50 n_{\mathrm{W}^{50+}}}{n_e} = 1 - 50 \cdot 10^{-5} = 1 - 0.0005 = 0.9995. \tag{24}$$

Therefore, the fractional reduction of the alpha particle heating power is:

$$\frac{P_\alpha^{\mathrm{diluted}}}{P_\alpha^{\mathrm{pure}}} = (0.9995)^2 = 0.999. \tag{25}$$

The reduction due to dilution is about 0.1 %. The radiative power loss density due to tungsten is:

$$P_{\mathrm{rad,\ Z}} = n_{\mathrm{W}^{50+}} n_e R_W = 10^{-5} \cdot 10^{20} \cdot 10^{20} \cdot 10^{-31} \frac{\mathrm{W}}{\mathrm{m}^3} = 10\,\mathrm{kW\,m^{-3}}. \tag{26}$$

This equals about 10 % of the undiluted alpha particle heating power.

(c) **Conclusions:** The calculations show that for low charge state impurities, the fusion performance is reduced both due to fuel dilution and radiation losses, whereas for high charge state impurities, the dominant performance reduction mechanism is the strong increase of radiative power losses.

Exercise 4.
The Lawson criterion and fuel dilution and impurity radiation.

For realistic approximation of the Lawson criterion fuel dilution and impurity radiation must be considered when evaluating the operating space igniting the plasma. To investigate these effects, download the .m files in the exercise repository of MyCourses. The main script plotting_script_for_ntaue.m plots the viable ignition parameter regime for given helium confinement time parameter ρ^*, impurity concentration f_Z, and impurity charge state Z, with a switch whether to include radiation losses or not. As the rate of fusion reactions and the helium concentration are coupled, the function solve_fhe.m solves the helium concentration f_{He} from equation (2.8) in [2].

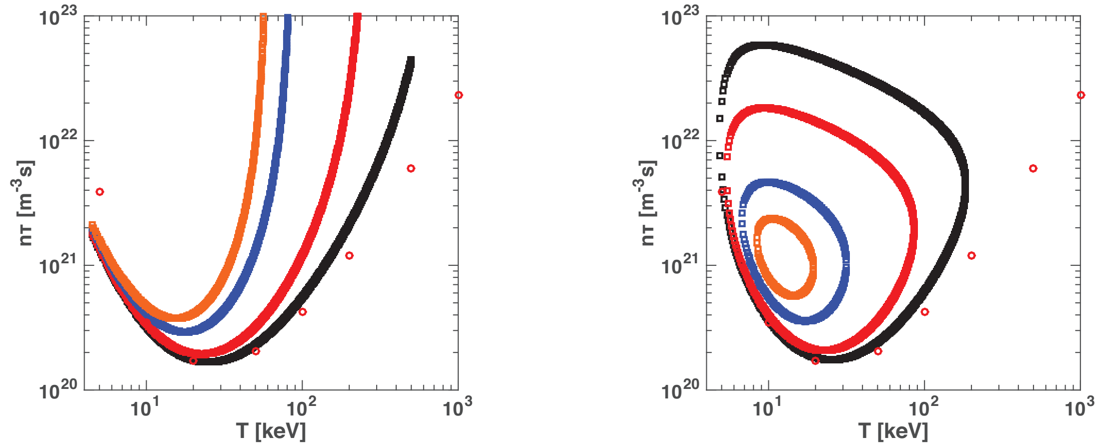
(a) Investigate the parameter space for ignition for a pure plasma (no impurities, $f_Z = 0$). Try running the script with and without the radiation contribution for different values of ρ^*. What drives the closure of the ignition domain for a pure plasma?

(b) Investigate the impact of carbon ($Z = 6$) and tungsten ($Z = 74$) impurities on the size of the ignition domain. Run the plotting script with $\rho^* = 5$ and try to replicate the figures in the Fusion Principles lecture slides . The result should look similar, but the models applied are necessarily not identical. Assume that the carbon impurities are fully ionized, and that the dominant charge state for the tungsten impurities is about 50. How do your results compare to the calculations in the previous exercise?

Solution 4.

(a) Radiation losses are required to reach the closure of the ignition domain, as can be observed in Fig. 3. However, the closure occurs also only if helium dilution is taken into account and connected to the energy confinement time. Both radiation losses and helium dilution are needed for the closure. The upper boundary in the ignition domain represents the location in the parameter space, where the fusion power is reduced below the Bremsstrahlung radiation power due to fuel dilution by helium (Fig. 4). In the example case, the dilution value is about 27%.

(b) The impact of impurities on the ignition contour ($\rho = 5$) can be found in Fig. 5. The left plot represents carbon contamination with $f_Z = 0, 1e-2, 2e-2, 3e-2$. The right plot represent tungsten contamination with $f_Z = 0, 4e-5, 8e-5$, and $1e-4$, assuming $Z = 74$



(a) (b)

Figure 3: Ignition contours for $\rho = 1$, 3, 9, and 13. The radiated power losses are neglected in the left plot. In the right plot radiated power losses are included. The circles represent pure DT plasma cases and include radiation in both cases.

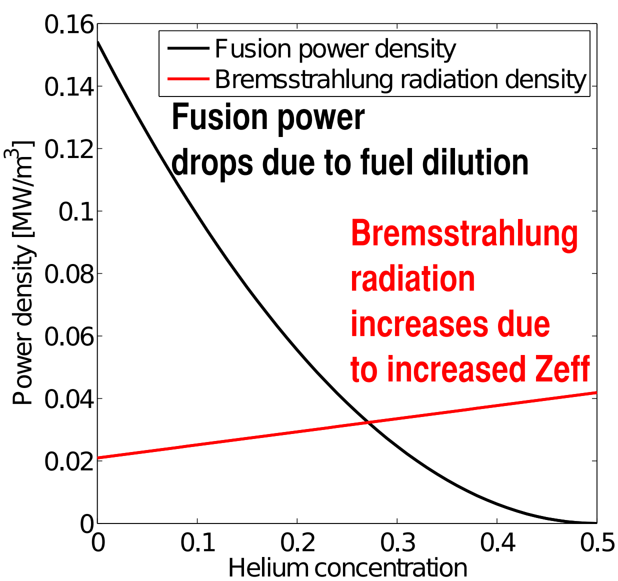
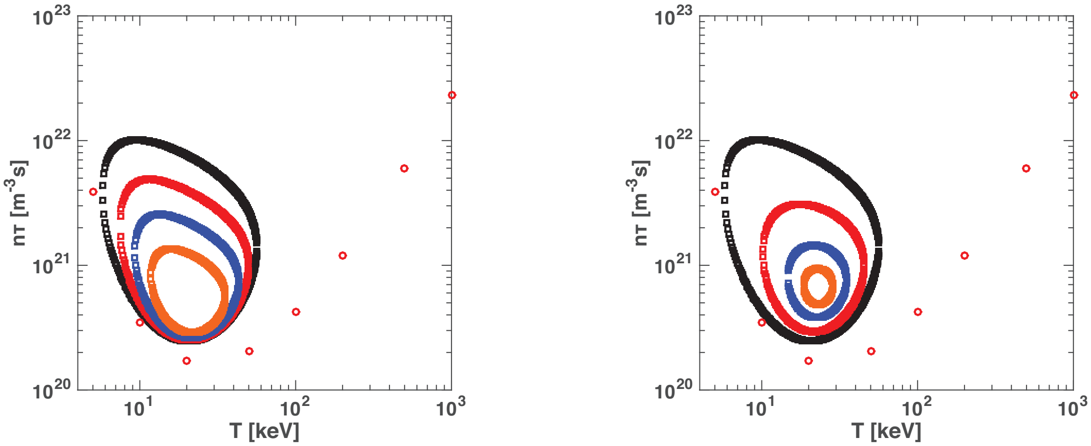


Figure 4: Fusion power density and the Bremsstrahlung radiation density as a function of the plasma helium concentration at $T = 15$ keV, $n = 10^{20}$ m^{-3}, and $\langle \sigma v \rangle \sim 1.1e - 22$ m^3/s



(a) (b)

Figure 5: The impact of impurities on the ignition contour ($\rho = 5$). The left plot represents carbon contamination with $f_Z = 0, 1e-2, 2e-2, 3e-2$. The right plot represent tungsten contamination with $f_Z = 0, 4e-5, 8e-5$, and $1e-4$, assuming $Z = 74$

Constants:

1 eV $= 1.602 \times 10^{-19}$ J

$m_p = 1.673 \times 10^{-27}$ kg

$m_n = 1.675 \times 10^{-27}$ kg

$N_A = 6.022 \times 10^{23}$ mol^{-1}

$k_B = 1.381 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

$\sigma_{SB} = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}

$c_B = 1.71 \times 10^{-38}$ W m^3 eV$^{-1/2}$

Power equations assuming pure hydrogenic (Z=1) plasma:

Bremsstrahlung radiation: $P_{Br} = c_B n_e^2 T_e^{1/2}$

Black body radiation: $P_{bb} = \sigma_{SB} T_S^4 A$

Fusion power density: $P_f = \alpha n_i n_j \langle \sigma v \rangle E_f$,

where E_f represents the produced energy per a fusion reaction, n_i and n_j are the fuel isotope densities, n_e the electron density, T_S the surface temperature of the black body, and A the surface area of the black body. The α parameter in the fusion power density equation is 1 for D-T fusion, and 1/2 for D-D fusion. This parameter arises due to the fact that when calculating the fusion cross-section integral ($\langle \sigma v \rangle$) for like particle collisions (D-D), every collision is counted twice. This should not be confused with the 1/4-factor that arises in the D-T fusion cross-section with 50-50 % fuel mixture due to $n_D = n_T = n_e/2 \rightarrow n_D \times n_T = (1/4) \times n_e^2$. More information can be found in e.g. [3].

References:

[1] J. Wesson, Tokamaks 3rd or 4th edition, Chapter 1.9

[2] D. Reiter, G.H. Wolf, H. Kever, "Burn condition, helium particle confinement and exhaust efficiency", Nucl. Fusion, 30, (1990), 2141
[3] J. P. Freidberg, Plasma Physics and Fusion Energy, Cambridge University Press, 2007, p.44