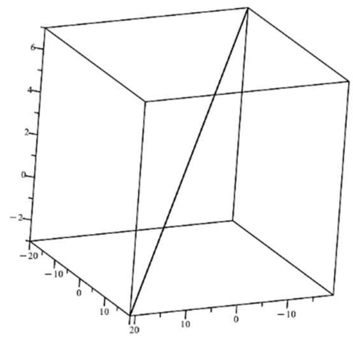
WEEK 1 – CLASS EXERCISES

EXERCISE 1: Find the equation of the line passing though the points $\boldsymbol{A} = (\mathbf{1}, \mathbf{0}, \mathbf{2})$ and $\boldsymbol{B} = (\mathbf{5}, \mathbf{4}, \mathbf{1})$. Sketch the line.

We can start by calculating the vector between the two points: $\overrightarrow{AB} = (5-1, 4-0, 1-2) = (4, 4, -1)$

Let's now use the vector equation of the line using the point A and the vector $\overrightarrow{AB}$:

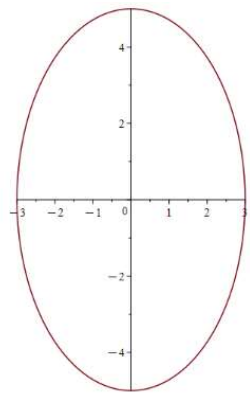
$$r(\lambda) = A + \lambda\, \overrightarrow{AB} = (1, 0, 2) + \lambda\, (4, 4, -1) = (1 + 4\lambda, 4\lambda, 2 - \lambda)$$



We can now plot the two points in the 3D space and sketch the line by hand. Otherwise, using Maple: `plot3d([1+4t, 4t, 2-t], t = -5..5)`

EXERCISE 2: Sketch the following curves:

$(\boldsymbol{a})$ $\boldsymbol{x(t) = 3\cos(t)}$, $\boldsymbol{y(t) = 5\sin(t)}$ for $\boldsymbol{0 \le t < 2\pi}$. What is this curve called?

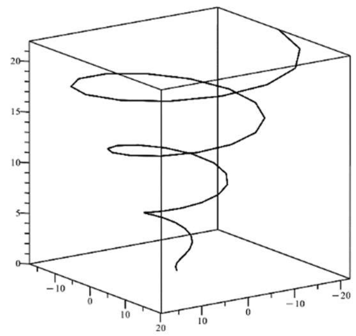


We see that the x coordinate oscillates between -3 and 3, whereas the y coordinate oscillates between -5 and 5. Using the identity $\cos^2(t) + \sin^2(t) = 1$, we can eliminate the parameter t and we get the following equation, which corresponds to an ellipse:

$$\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1 \quad \Longrightarrow \quad \frac{x^2}{9} + \frac{y^2}{25} = 1$$

We can try to locate certain points in order to sketch it by hand. Otherwise, using Maple: `plot([3cos(t), 5sin(t), t = 0 .. 2π])`

$(\boldsymbol{b})$ $\boldsymbol{x(t) = t\cos(t)}$, $\boldsymbol{y(t) = t\sin(t)}$, $\boldsymbol{z(t) = t}$ for $\boldsymbol{0 \le t \le 7\pi}$.

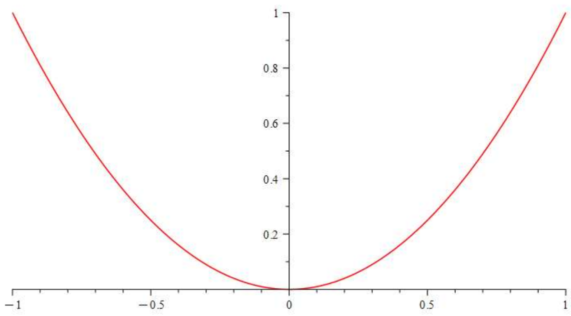


The x and y coordinates resemble to the parametric equation of the circle. However, because they both have the factor t, the radius increases as t increases, that is, we have a spiral. Moreover, the z coordinate translates the spiral along the z axis, forming a cone. All this information and with the help of some points, it should be enough to sketch the curve by hand. Otherwise, using Maple: `plot3d([t*cos(t), t*sin(t), t], t = 0 .. 7π)`.

EXERCISE 3: Consider the parametric curve $\boldsymbol{x(t) = \cos(t)}$, $\boldsymbol{y(t) = \cos^2(t)}$ for $-\infty < \boldsymbol{t} < \infty$.

$(\boldsymbol{a})$ Sketch the curve and carefully describe the motion. Think carefully about the range of $\boldsymbol{x(t)}$ and $\boldsymbol{y(t)}$.

We first have to realise that the values of $x(t)$ oscillate between -1 and 1, and the values for $y(t)$ oscillate between 0 and 1. Giving different values to the parameter t, we realise that the curve corresponds to a parabola, which oscillates back and forth. The motion is repeated over and over again. We could sketch the graph of the curve by hand by giving values to t and locating the points. Otherwise, using Maple: `plot([cos(t), cos²(t), t = 0 .. π])`.

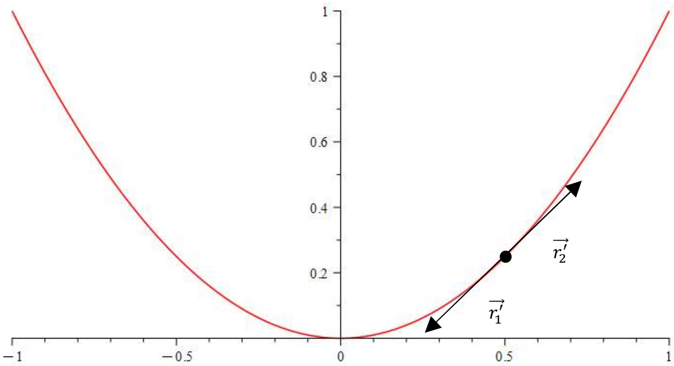


(b) Find the tangent vectors at the point $A = (1/2, 1/4)$. Make a sketch and relate your answers to the direction of motion.

The tangent vector to the curve at any point will be given by:

$$\overrightarrow{r'}(t) = \left(x'(t), y'(t)\right) = (-\sin(t), -2\cos(t)\sin(t)) = (-\sin(t), -\sin(2t))$$

We need to find the value of the parameter t for the point A. We have that $\cos(t) = 1/2$, which means that t is either $\pi/3 + 2\pi k$ or $-\pi/3 + 2\pi k$, where $k \in \mathbb{Z}$. By direct substitution, in the first case, for $t_1 = \pi/3 + 2\pi k$, the tangent vector is $\overrightarrow{r_1'} = \left(-\sqrt{3}/2, -\sqrt{3}/2\right)$, which means that the motion of the parabola goes towards the left. On the second case, for $t_2 = -\pi/3 + 2\pi k$, the tangent vector is $\overrightarrow{r_1'} = \left(\sqrt{3}/2, \sqrt{3}/2\right)$ and the motion goes towards the right.



(c) Find the tangent vector at the point $B = (1, 1)$. Does your answer make sense? Is the curve smooth at this point?

In this case, the parameter t will have the value $t = 2\pi k$, with $k \in \mathbb{Z}$. Using the formula for tangent vector that we derived in question (b), we get that $\overrightarrow{r'}(2\pi k) = (0, 0)$, which gives no information about the slope of the curve, since the curve is not smooth at this point, and it cannot have a tangent vector. When the curve gets to this point, it stops and changes direction. Thus, this vector indicates the velocity of the motion, which is indeed a smooth function.

(d) Find the length of the curve.

Since the curve is being overwritten infinitely many times, we only need to calculate the length of the curve that goes from point $(-1, 1)$ to $(1, 1)$, that is, when t is, for example, between 0 and π:

$$\int_0^{\pi} \sqrt{\sin^2(t) + 4\sin^2(t)\cos^2(t)}\ dt \;=\; \int_0^{\pi} |\sin(t)|\,\sqrt{1 + 4\cos^2(t)}\ dt$$

$$= \int_0^{\pi} \sin(t)\ \sqrt{1 + 4\cos^2(t)}\ dt \qquad since\ \sin(t) \geq 0\ on\ [0, \pi]$$

$$= -\frac{1}{2}\int_{-2}^{2} \sqrt{1 + x^2}\, dx \qquad let\ x = 2\cos(t)\ \ |\ \ dx = -2\sin(t)\ dt$$

$$= \frac{1}{2}\int_{-2}^{2} \sqrt{1 + x^2}\, dx \qquad changing\ the\ limits\ of\ integration$$

$$= \int_0^{2} \sqrt{1 + x^2}\, dx \qquad because\ the\ function\ is\ even$$

In order to solve the integral above, let's do a change of variables. Let $x = \tan(s)$ and also $dx = \sec^2(s)\,ds$. Also remember that $1 + \tan^2(s) = \sec^2(s)$

$$\int \sqrt{1 + x^2}\, dx = \int \sec^2(s)\ \sqrt{1 + \tan^2(s)}\ ds = \int \sec^3(s)\ ds$$

We now need to integrate by parts. Let $u = \sec(s)$ and $dv = \sec^2(s)\ ds$. We then have $du = \sec(s)\tan(s)\ ds$ and $v = \tan(s)$:

$$\int \sec^3(s)\ ds = \sec(s)\tan(s) - \int \sec(s)\tan^2(s)\ ds$$

We can now write $\tan^2(s) = \sec^2(s) - 1$:

$$\int \sec^3(s)\ ds = \sec(s)\tan(s) - \int \sec(s)\,(\sec^2(s) - 1)\ ds = \sec(s)\tan(s) - \int \sec^3(s)\ ds + \int \sec(s)\ ds$$

and rearranging:

$$2\int \sec^3(s)\ ds = \sec(s)\tan(s) + \int \sec(s)\ ds \quad \Longrightarrow \quad \int \sec^3(s)\ ds = \frac{1}{2}\sec(s)\tan(s) + \frac{1}{2}\int \sec(s)\ ds$$

We now need to solve the integral of $\sec(s)$. In order to do that, multiply and divide by $\sec(s) + \tan(s)$:

$$\int \sec(s)\ ds = \int \frac{\sec(s)\,(\sec(s) + \tan(s))}{\sec(s) + \tan(s)}\ ds = \int \frac{\sec^2(s) + \sec(s)\tan(s)}{\sec(s) + \tan(s)}\ ds$$

Another change of variables: Let $h = \sec(s) + \tan(s)$, and therefore, $dh = (\sec(s)\tan(s) + \sec^2(s))\,ds$:

$$\int \sec(s)\ ds = \cdots = \int \frac{\sec^2(s) + \sec(s)\tan(s)}{\sec(s) + \tan(s)}\ ds = \int \frac{1}{h}\ dh = \ln|h| = \ln|\sec(s) + \tan(s)|$$

Retracing our steps:

$$\int \sec^3(s)\ ds = \frac{1}{2}\sec(s)\tan(s) + \frac{1}{2}\int \sec(s)\ ds = \frac{1}{2}\sec(s)\tan(s) + \frac{1}{2}\ln|\sec(s) + \tan(s)|$$

Now, remembering that $x = \tan(s)$ and $\sqrt{1 + x^2} = \sec(s)$:

$$\int \sqrt{1+x^2}\, dx = \int \sec^3(s)\, ds = \frac{1}{2}\sec(s)\tan(s) + \frac{1}{2}\ln|\sec(s)+\tan(s)| = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left| x + \sqrt{1+x^2} \right|$$

Finally, the arc length of the curve is:

$$\int_0^2 \sqrt{1+x^2}\, dx = \left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left| x+\sqrt{1+x^2}\right|\right]_0^2 = \sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5}) - \frac{1}{2}\ln(1) = \sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5}) \approx 2.958\, u$$

EXERCISE 4: Consider the curve of intersection of the plane $\boldsymbol{z = y}$ and the parabolic cylinder $\boldsymbol{y = 4 - x^2}$.

($\boldsymbol{a}$) Find a parametric equation $\boldsymbol{r(t) = (x(t), y(t), z(t))}$ of the curve.

We can parametrise the intersection curve by letting $x(t) = t$. Our parametrised curve will then be:

$$r(t) = (t, 4-t^2, 4-t^2)$$

($\boldsymbol{b}$) Find the arc length of the part of the curve that lies above the $\boldsymbol{xy}$-plane.

As we only want to consider the curve above the xy-plane, we will only consider the interval where $-2 \le t \le 2$. The modulus of the derivative vector is:

$$|\vec{r}\,'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2} = \sqrt{1^2 + (-2t)^2 + (-2t)^2} = \sqrt{1+8t^2}$$

and therefore, the arc length of the curve is calculated as:

$$\int_{-2}^{2} |\vec{r}\,'(t)|\, dt = \int_{-2}^{2} \sqrt{1+8t^2}\, dt = 2\int_0^2 \sqrt{1+8t^2}\, dt$$

where the last equality comes from realising that the function is even. We can solve this integral the same way as in the previous exercise. However, in this case let's use hyperbolic functions to learn another method. Let's do the following change of variables: Let $\sqrt{8}t = \sinh(u)$ and therefore $\sqrt{8}\, dt = \cosh(u)\, du$. Also remember that $\cosh^2(u) - \sinh^2(u) = 1$:

$$\int \sqrt{1+8t^2}\, dt = \int \sqrt{1+\sinh^2(u)}\, \frac{\cosh(u)}{\sqrt{8}}\, du = \int \cosh(u)\, \frac{\cosh(u)}{\sqrt{8}}\, du = \frac{1}{\sqrt{8}} \int \cosh^2(u)\, du$$

We can write the hyperbolic cosine as:

$$\cosh(u) = \frac{e^u + e^{-u}}{2}$$

and therefore:

$$\frac{1}{\sqrt{8}} \int \cosh^2(u)\, du = \frac{1}{2\sqrt{2}} \int \left(\frac{e^u + e^{-u}}{2}\right)^2 du = \frac{1}{8\sqrt{2}} \int (e^{2u} + 2 + e^{-2u})\, du = \frac{1}{8\sqrt{2}}\left(\frac{e^{2u}}{2} + 2u - \frac{e^{-2u}}{2}\right)$$

Now, remembering that $\sinh(2u) = 2\sinh(u)\cosh(u)$:

$$\frac{1}{8\sqrt{2}}\left(\frac{e^{2u}}{2} + 2u - \frac{e^{-2u}}{2}\right) = \frac{1}{8\sqrt{2}}(2u + \sinh(2u)) = \frac{1}{8\sqrt{2}}(2u + 2\sinh(u)\cosh(u))$$

Retracing our steps and, once again, keeping into account that $\cosh^2(u) - \sinh^2(u) = 1$:

$$\frac{1}{\sqrt{8}}\int \cosh^2(u)\ du = \frac{1}{4\sqrt{2}}\left(u + \sinh(u)\sqrt{1+\sinh^2(u)}\right)$$

Now, since $\sinh(u) = \sqrt{8}t$, then $u = \operatorname{arcsinh}(\sqrt{8}t) = \ln(\sqrt{8}t + \sqrt{1+8t^2})$. That is:

$$\int \sqrt{1+8t^2}\ dt = \frac{1}{\sqrt{8}}\int \cosh^2(u)\ du = \frac{1}{4\sqrt{2}}\left(u + \sinh(u)\sqrt{1+\sinh^2(u)}\right) = \frac{1}{4\sqrt{2}}\left(\ln\left(\sqrt{8}t + \sqrt{1+8t^2}\right) + t\sqrt{8+64t^2}\right)$$

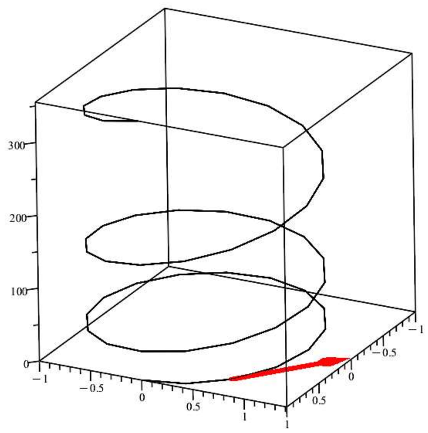
Finally, the arc length will be:

$$2\int_0^2 \sqrt{1+8t^2}\ dt = \frac{1}{2\sqrt{2}}\left[\ln\left(\sqrt{8}t + \sqrt{1+8t^2}\right) + t\sqrt{8+64t^2}\right]_0^2 = \frac{1}{2\sqrt{2}}\left(\ln(4\sqrt{2}+\sqrt{33}) + 4\sqrt{66}\right) \approx 12.3496\ u$$

EXERCISE 5: Consider the curve with parametric equations $\boldsymbol{r(t)} = (\boldsymbol{\cos(t)}, \boldsymbol{\sin(t)}, \boldsymbol{t^2})$ for $\boldsymbol{0 \le t \le 6\pi}$.

$(\boldsymbol{a})$ Sketch the curve and the tangent vector to the curve when $\boldsymbol{t = \pi/4}$.

We have a unit circle in the xy-plane that is being translated along the z axis. We can try to locate certain points in order to sketch it by hand. Otherwise, using Maple and then drawing the tangent vector:




```
with(plots)
with(plottools)
a := plot3d([cos(t), sin(t), t²], t = 0 .. 6π)
b := arrow( ⟨√2/2, √2/2, π²/16⟩, ⟨−√2/2, √2/2, π/2⟩, colour = red )
```

$(\boldsymbol{b})$ Compute the tangent vector at $\boldsymbol{t = \pi/4}$. Does your sketch match the computation?

$$\vec{r'}(t) = (x'(t), y'(t), z'(t)) = (-\sin(t), \cos(t), 2t)$$

By substituting $t = \pi/4$, we get that:

$$\vec{r'}(t) = \left(-\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{\pi}{4}\right), 2 * \frac{\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{2}\right)$$

The direction of such vector seems to match with that of our drawing.

$(\boldsymbol{c})$ Compute the arc length of the curve.

The modulus of the derivative vector is:

$$|\vec{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{\sin^2(t) + \cos^2(t) + (2t)^2} = \sqrt{1+4t^2}$$

and therefore, the arc length of the curve is calculated as:

$$\int_0^{6\pi} \sqrt{1+4t^2}\, dt$$

In order to solve the integral above, let $2t = x$ and therefore $2\, dt = dx$:

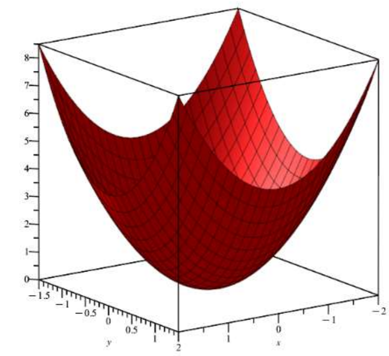
$$\int_0^{6\pi} \sqrt{1+4t^2}\, dt = \frac{1}{2}\int_0^{12} \sqrt{1+x^2}\, dx$$

We already calculated how to solve this integral in exercise 3d, so the arc length is:

$$\frac{1}{2}\int_0^{12\pi} \sqrt{1+x^2}\, dx = \frac{1}{2}\left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\right]_0^{12\pi} = \frac{1}{2}\left(6\pi\sqrt{1+144\pi^2} + \frac{1}{2}\ln\left|12\pi + \sqrt{1+144\pi^2}\right|\right) =$$

$$= 3\pi\sqrt{1+144\pi^2} + \frac{1}{4}\ln\left(12\pi + \sqrt{1+144\pi^2}\right) \approx 356.511\, u$$

EXERCISE 6: Consider the function $\boldsymbol{f(x,y) = x^2 + 2y^2}$

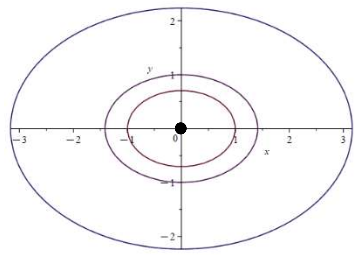


$(\boldsymbol{a})$ Sketch the graph of $\boldsymbol{f(x,y)}$. That is, the surface determined by $\boldsymbol{z = f(x,y)}$.

This is a paraboloid with elliptical cross-section determined by the equation $x^2 + 2y^2 = c$ for any constant $c > 0$. We can use Maple in order to plot it:

```
plot3d(x² + 2y²)
```

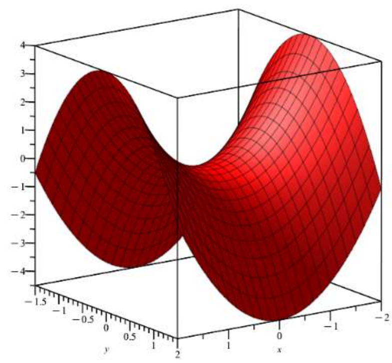
$(\boldsymbol{b})$ Find and sketch the level curves $\boldsymbol{f = -1}$, $\boldsymbol{f = 0}$, $\boldsymbol{f = 1}$, $\boldsymbol{f = 2}$ and $\boldsymbol{f = 10}$.



The level curves are ellipses with equation $x^2 + 2y^2 = c$ for any constant $c > 0$. The level curve $f = -1$ is thus empty. Here's a plot of the other four level curves:

```
with(plots)
contourplot(x² + 2y², contours = [−1, 0, 1, 2, 10])
```

EXERCISE 7: Consider the function $\boldsymbol{f(x,y) = x^2 - 2y^2}$



$(\boldsymbol{a})$ Sketch the graph of $\boldsymbol{f(x,y)}$. That is, the surface determined by $\boldsymbol{z = f(x,y)}$.

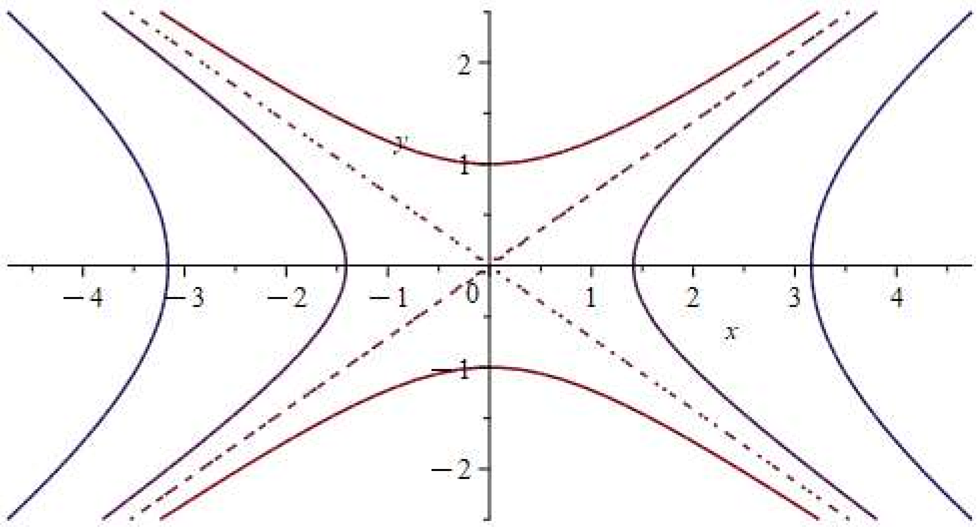
This is a hyperboloid with hyperbolic cross-section determined by the equation $x^2 - 2y^2 = c$ for any constant c. We can use Maple in order to plot it:

```
plot3d(x² − 2y²)
```

$(\boldsymbol{b})$ Find and sketch the level curves $\boldsymbol{f = -2}$, $\boldsymbol{f = 0}$, $\boldsymbol{f = 2}$ and $\boldsymbol{f = 10}$.

The level curves are hyperbolas with equation $x^2 - 2y^2 = c$ for any constant c:

```
with(plots)
contourplot(x² − 2y², contours = [−2, 0, 2, 10])
```



EXERCISE 8: You are given two sets of level curves: one for a cone and one for a paraboloid. Which one is which? Justify your answer.

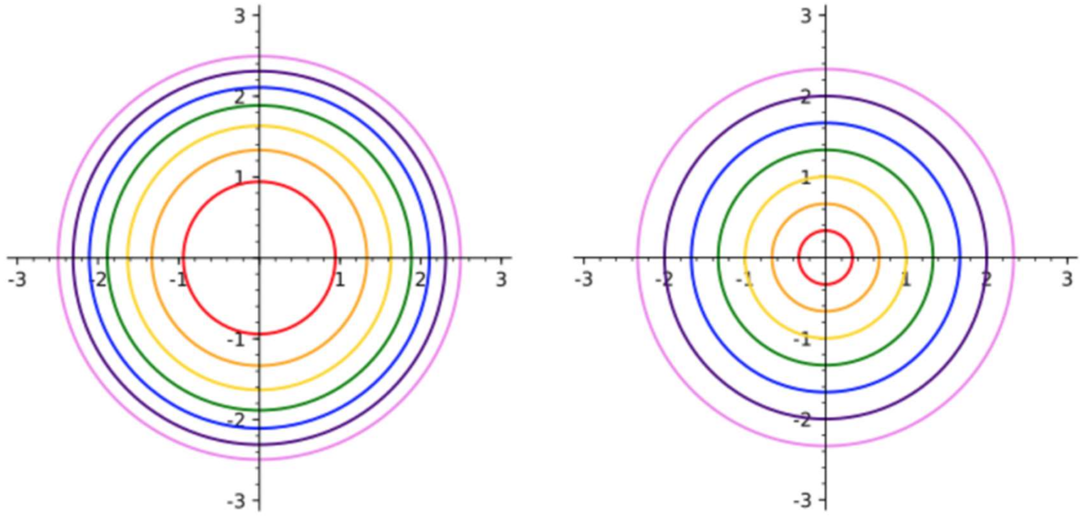


Figure 1: exercise 14.1.7 in Guichard's Calculus text.

The one on the left corresponds to a paraboloid, since the level curves are closer and closer each time, which means that the slope of the function is increasing. On the other hand, the one on the right represents a cone because the level curves are evenly separated. That is, the level curves could correspond respectively to the following functions: