WEEK 2 – CLASS EXERCISES

EXERCISE 1: Compute the following limits or show that they do not exist.

(a) Let us try with the line $y = mx$, where m is a parameter that indicates the slope of the line:

$$\lim_{(x,y)\to(0,0)} \frac{y^2 - x^2}{x^2 + y^2} = \lim_{x\to 0} \frac{(mx)^2 - x^2}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{m^2x^2 - x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2\,(m^2 - 1)}{x^2\,(1 + m^2)} = \lim_{x\to 0} \frac{m^2 - 1}{m^2 + 1} = \frac{m^2 - 1}{m^2 + 1}$$

The limit depends on the slope m, that is, it depends on the line along which we are moving, resulting in different results with each different line. Therefore, the limit does not exist.

(b) Let us try with functions of the form $y = ax^3$, where a is a parameter that determines the shape of the function:

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^3\,(ax^3)}{x^6 + (ax^3)^2} = \lim_{x\to 0} \frac{ax^6}{x^6 + a^2x^6} = \lim_{x\to 0} \frac{ax^6}{x^6\,(1 + a^2)} = \lim_{x\to 0} \frac{a}{1 + a^2} = \frac{a}{1 + a^2}$$

The limit depends on the parameter a, that is, it depends on the function along which we are moving, resulting in different results with each different function. Therefore, the limit does not exist.

(c) We can try to solve the limit from some paths y and the result will be 0. This way, we can realise that the limit probably exists (as it is always the same), but we will have to prove it. We can use polar coordinates to solve this limit by making $r \to 0$ from every direction, that is, independently of the angle θ. Therefore:

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2 + y^2} = \lim_{r\to 0^+} \frac{(r\sin\theta)^4}{(r\cos\theta)^2 + (r\sin\theta)^2} = \lim_{r\to 0^+} \frac{r^4\sin^4\theta}{r^2\,(\cos^2\theta + \sin^2\theta)} = \lim_{r\to 0^+} (r^2\sin^4\theta) = 0$$

Note that the limit will be 0 independently of the angle θ.

REMARK! Be careful when using polar coordinates. For instance, in part (b):

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3\,(r\sin\theta)}{(r\cos\theta)^6 + (r\sin\theta)^2} = \lim_{r\to 0^+} \frac{r^4\cos^3\theta\,\sin\theta}{r^2\,(r^4\cos^6\theta + \sin^2\theta)} = \lim_{r\to 0^+} \frac{r^2\cos^3\theta\,\sin\theta}{r^4\cos^6\theta + \sin^2\theta}$$

The above limit is not zero since, depending on the path, r and θ can be related in such way that we obtain different limits.

- If θ is constant (along lines): $\lim_{r\to 0^+} \frac{r^2\cos^3\theta\ \text{si}}{r^4\cos^6\theta+\sin^2\theta} = 0$
- If $y = x^3$ then $r\sin\theta = r^3\cos^3\theta \implies$ Relation between r and θ: $\sin\theta = r^2\cos^3\theta$

$$\lim_{r\to 0^+} \frac{(\sin\theta)\,(\sin\theta)}{\sin^2\theta + \sin^2\theta} = \lim_{r\to 0^+} \frac{\sin^2\theta}{2\sin^2\theta} = \lim_{r\to 0^+} \frac{1}{2} = \frac{1}{2}$$

Another way of solving the limit in part (c) would have been by applying the squeeze theorem. In order to do so, we first need to find some upper and lower bounds for the function. When $y \neq 0$:

$$\frac{y^4}{x^2+y^2} \leq \frac{y^4}{y^2} \leq y^2$$

Also, when $y = 0$:

$$\frac{y^4}{x^2+y^2} = \frac{0^4}{x^2+0^2} = 0 \leq y^2$$

That is, y^2 will work as an upper bound. We can easily find a lower bound for the function by realising that it is always positive, that is, $\mathbf{0}$ is a lower bound for the function. Therefore:

$$0 \leq \frac{y^4}{x^2+y^2} \leq y^2$$

Finally, since $\lim_{(x,y)\to(0,0)} 0 = 0$ and $\lim_{(x,y)\to(0,0)} y^2 = 0$, then the limit of our function exists and is also $\mathbf{0}$. That is:

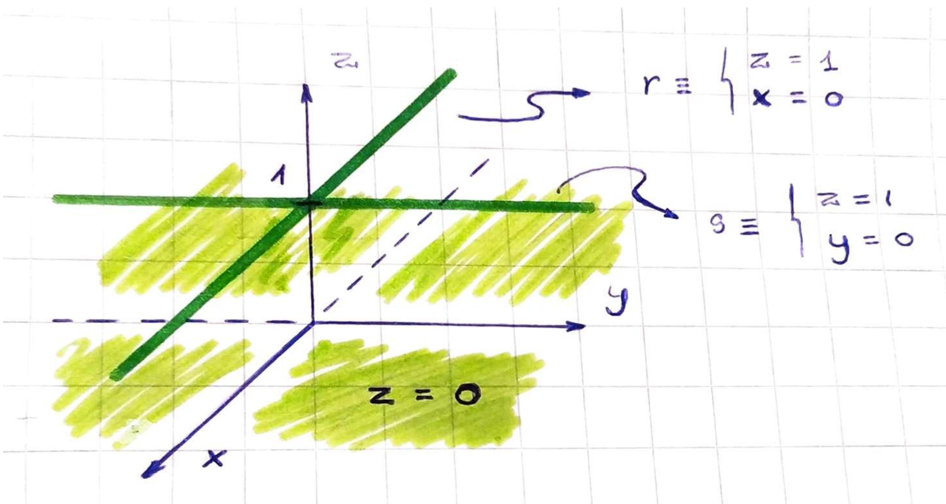
$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = 0$$

EXERCISE 2: Consider the function $\boldsymbol{f(x,y)}$ defined by:

$$f(x,y) = \begin{cases} \mathbf{1} & \boldsymbol{if\ x = 0\ or\ y = 0} \\ \mathbf{0} & \boldsymbol{otherwise} \end{cases}$$

$(\boldsymbol{a})$ Sketch the surface $\boldsymbol{z = f(x,y)}$.

It is a plane in $z = 0$, except for the axes x and y, where the function has value 1:



$(\boldsymbol{b})$ Is the function continuous at $(\mathbf{0},\mathbf{0})$?

The value of the function at the point $(0,0)$ is $f(0,0) = 1$. If we approach the point $(0,0)$ along the x or y axes we indeed get that:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

However, if we approached such point from any other direction, we would get that:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Since the limit does not exist, the function is not continuous at the point $(0,0)$.

$(\boldsymbol{c})$ Compute the first partial derivatives at the point $(\mathbf{0},\mathbf{0})$. Note that due to the piecewise definition of the function, you must use definition of the partial derivatives in order to compute them.

$$\left.\frac{\partial}{\partial x} f(x,y)\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\left.\frac{\partial}{\partial y} f(x,y)\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$(\boldsymbol{d})$ Find the equation of the tangent plane you would get by just plugging the data from part $(\boldsymbol{c})$ in the tangent plane equation.

We know from the lectures that the equation for a tangent plane at some point (x_0, y_0) is the following:

$$z = f(x_0, y_0) + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$$

where f_x and f_y are the partial derivatives with respect to x and y respectively.

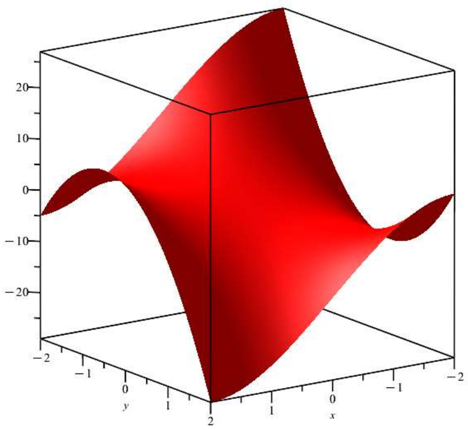
Substituting with the partial derivatives calculated in (c), the tangent plane at the point $(0,0)$ would be: $z = 1$

$(\boldsymbol{e})$ Does the plane you found in part $(\boldsymbol{d})$ approximate the surface well in any small region around $(\mathbf{0},\mathbf{0})$?

It only approximates the values along the x and y axes, so it is not a good linear approximation for the function.

$(\boldsymbol{f})$ Does the surface $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ have a tangent plane at $(\mathbf{0},\mathbf{0})$? Discuss / Explain.

No, because the function is not continuous.



EXERCISE 3: Consider the function $\boldsymbol{f(x,y) = x^3 - 3xy^2 - 6y - 1}$.

$(\boldsymbol{a})$ Compute all the first and second order partial derivatives.

$$\frac{\partial}{\partial x} f(x,y) = 3x^2 - 3y^2 \qquad \frac{\partial^2}{\partial x^2} f(x,y) = 6x$$

$$\frac{\partial}{\partial y} f(x,y) = -6xy - 6 \qquad \frac{\partial^2}{\partial y^2} f(x,y) = -6x$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}(3x^2 - 3y^2) = -6y$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \frac{\partial}{\partial y}(-6xy - 6) = -6y$$

As the function is twice differentiable and its derivatives are continuous, its second order mixed partial derivatives are equal.

(b) Find all the points on the surface where the tangent plane is horizontal.

The points on the surface of $z = f(x,y)$ with a horizontal tangent plane are those where both first-order partial derivatives are 0. We can therefore find those points with a system of equations:

$$\begin{cases} 3x^2 - 3y^2 = 0 \\ -6xy - 6 = 0 \end{cases} \Rightarrow \begin{cases} x^2 = y^2 \\ xy = -1 \end{cases} \Rightarrow \quad y = -\frac{1}{x} \quad \Rightarrow \quad x^2 = \frac{1}{x^2} \quad \Rightarrow \quad x^4 = 1 \quad \Rightarrow \quad x = \pm 1 \quad \Rightarrow \quad y = \mp 1$$

Therefore, the points where the tangent plane is horizontal are $(1,-1)$ and $(-1,1)$. We can substitute those values in the partial derivatives to check that our calculations where correct:

$$f_x(1,-1) = 3(1)^2 - 3(-1)^2 = 3 - 3 = 0 \quad and \quad f_y(1,-1) = -6(1)(-1) - 6 = 6 - 6 = 0$$

$$f_x(-1,1) = 3(-1)^2 - 3(1)^2 = 3 - 3 = 0 \quad and \quad f_y(-1,1) = -6(-1)(1) - 6 = 6 - 6 = 0$$

(c) Find the tangent plane to the surface at the point $(\mathbf{1},\mathbf{2})$.

Let's compute the value of the function and the first-order partial derivatives at the point $(1,2)$:

$$f(1,2) = (1)^3 - 3(1)(2)^2 - 6(2) - 1 = 1 - 3 * 4 - 6 * 2 - 1 = 1 - 12 - 12 - 1 = -24$$

$$f_x(1,2) = 3(1)^2 - 3(2)^2 = 3 - 3 * 4 = 3 - 12 = -9$$

$$f_y(1,2) = -6(1)(2) - 6 = -6 * 2 - 6 = -12 - 6 = -18$$

Finally, using the equation for a tangent plane at a point (x_0, y_0):

$$z = f(x_0,y_0) + f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0)$$

$$z = f(1,2) + f_x(1,2)\,(x-1) + f_y(1,2)\,(y-2) = -24 - 9\,(x-1) - 18\,(y-2) = -24 - 9x + 9 - 18y + 36$$

$$\Rightarrow \quad z = -9x - 18y + 21 \quad \Rightarrow \quad 9x + 18y + z = 21$$

EXERCISE 4: One of the most crucial operators in Partial Differential Equations and thus in, for example, models of quantum mechanics is the Laplacian of a continuous function $\boldsymbol{f(x,y)}$ with continuous second-order partial derivatives in $\mathbb{R}^2$. It is denoted by Δ or ∇^2 and defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

The *polar coordinates* $\boldsymbol{r}, \boldsymbol{\theta}$ of $\mathbb{R}^2$ are given by the change of variables:

$$(\boldsymbol{x}, \boldsymbol{y}) \to (\boldsymbol{r}\cos\boldsymbol{\theta}, \boldsymbol{r}\sin\boldsymbol{\theta})$$

for $\boldsymbol{r} > \mathbf{0}$ and $-\boldsymbol{\pi} < \boldsymbol{\theta} \leq \boldsymbol{\pi}$. Then we can always write $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ depending on $\boldsymbol{r}$ and $\boldsymbol{\theta}$ as follows:

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{r}\cos\boldsymbol{\theta}, \boldsymbol{r}\sin\boldsymbol{\theta})$$

Show that we have the following change of variable formula for the Laplacian of $\boldsymbol{f}$ in the polar coordinates:

$$\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{x}^2} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{y}^2} = \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{r}^2} + \frac{\mathbf{1}}{\boldsymbol{r}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{r}} + \frac{\mathbf{1}}{\boldsymbol{r}^2}\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{\theta}^2}$$

In order to show that the formula above holds, we need to remember the chain rule:

$$\frac{\partial}{\partial x} f\big(g(x,y), h(x,y)\big) = \frac{\partial f}{\partial g} * \frac{\partial g}{\partial x} + \frac{\partial f}{\partial h} * \frac{\partial h}{\partial x}$$

Let's start by observing that:

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta \qquad \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta \qquad \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta$$

Let's now compute the corresponding first- and second-order partial derivatives with respect to the variable r:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} * \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} * \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} * (\cos\theta) + \frac{\partial f}{\partial y} * (\sin\theta) = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial}{\partial r}\left(\cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y}\right) &= \cos\theta\left(\frac{\partial^2 f}{\partial x^2} * \frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y \partial x} * \frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial^2 f}{\partial x \partial y} * \frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2} * \frac{\partial y}{\partial r}\right) \\
&= \cos\theta\left(\frac{\partial^2 f}{\partial x^2} * (\cos\theta) + \frac{\partial^2 f}{\partial y \partial x} * (\sin\theta)\right) + \sin\theta\left(\frac{\partial^2 f}{\partial x \partial y} * (\cos\theta) + \frac{\partial^2 f}{\partial y^2} * (\sin\theta)\right) \\
&= \cos^2\theta\,\frac{\partial^2 f}{\partial x^2} + \sin^2\theta\,\frac{\partial^2 f}{\partial y^2} + 2\cos\theta\sin\theta\,\frac{\partial^2 f}{\partial x \partial y}
\end{aligned}$$

and now the partial derivatives with respect to θ:

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} * \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} * \frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x} * (-r\sin\theta) + \frac{\partial f}{\partial y} * (r\cos\theta) = -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) &= \frac{\partial}{\partial \theta}\left(-r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}\right) \\
&= -r\cos\theta\,\frac{\partial f}{\partial x} - r\sin\theta\left(\frac{\partial^2 f}{\partial x^2} * \frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y \partial x} * \frac{\partial y}{\partial \theta}\right) - r\sin\theta\,\frac{\partial f}{\partial y} + r\cos\theta\left(\frac{\partial^2 f}{\partial x \partial y} * \frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2} * \frac{\partial y}{\partial \theta}\right) \\
&= -r\cos\theta\,\frac{\partial f}{\partial x} - r\sin\theta\left(\frac{\partial^2 f}{\partial x^2} * (-r\sin\theta) + \frac{\partial^2 f}{\partial y \partial x} * (r\cos\theta)\right) - r\sin\theta\,\frac{\partial f}{\partial y} \\
&\quad + r\cos\theta\left(\frac{\partial^2 f}{\partial x \partial y} * (-r\sin\theta) + \frac{\partial^2 f}{\partial y^2} * (r\cos\theta)\right) \\
&= -r\cos\theta\,\frac{\partial f}{\partial x} - r\sin\theta\,\frac{\partial f}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 f}{\partial x^2} + r^2\cos^2\theta\,\frac{\partial^2 f}{\partial y^2} - 2r^2\sin\theta\cos\theta\,\frac{\partial^2 f}{\partial x \partial y}
\end{aligned}$$

Finally, we can now just plug in the partial derivatives we have just calculated:

$$
\begin{aligned}
&\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \\
&\qquad = \cos^2\theta \, \frac{\partial^2 f}{\partial x^2} + \sin^2\theta \, \frac{\partial^2 f}{\partial y^2} + 2\cos\theta\sin\theta \, \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{r}\left(\cos\theta \, \frac{\partial f}{\partial x} + \sin\theta \, \frac{\partial f}{\partial y}\right) \\
&\qquad + \frac{1}{r^2}\left(-r\cos\theta \, \frac{\partial f}{\partial x} - r\sin\theta \, \frac{\partial f}{\partial y} + r^2\sin^2\theta \, \frac{\partial^2 f}{\partial x^2} + r^2\cos^2\theta \, \frac{\partial^2 f}{\partial x^2} - 2r^2\sin\theta\cos\theta \, \frac{\partial^2 f}{\partial x \partial y}\right) \\
&\qquad = \cos^2\theta \, \frac{\partial^2 f}{\partial x^2} + \sin^2\theta \, \frac{\partial^2 f}{\partial y^2} + 2\cos\theta\sin\theta \, \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{r}\cos\theta \, \frac{\partial f}{\partial x} + \frac{1}{r}\sin\theta \, \frac{\partial f}{\partial y} - \frac{1}{r}\cos\theta \, \frac{\partial f}{\partial x} \\
&\qquad - \frac{1}{r}\sin\theta \, \frac{\partial f}{\partial y} + \sin^2\theta \, \frac{\partial^2 f}{\partial x^2} + \cos^2\theta \, \frac{\partial^2 f}{\partial y^2} - 2\sin\theta\cos\theta \, \frac{\partial^2 f}{\partial x \partial y} \\
&\qquad = (\cos^2\theta + \sin^2\theta) \, \frac{\partial^2 f}{\partial x^2} + (\cos^2\theta + \sin^2\theta) \, \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \qquad \blacksquare
\end{aligned}
$$