WEEK 3 – CLASS EXERCISES

EXERCISE 1: Suppose that at the point $(\mathbf{1},\mathbf{2})$ the directional derivative of the function $\boldsymbol{z}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ in the direction $\mathbf{3}\vec{\boldsymbol{\imath}}+\mathbf{4}\vec{\boldsymbol{\jmath}}$ is equal to $\mathbf{2}$ and the directional derivative of $\boldsymbol{z}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ in the direction $\mathbf{5}\vec{\boldsymbol{\imath}}+\mathbf{12}\vec{\boldsymbol{\jmath}}$ is equal to $\mathbf{1}$.

$(\boldsymbol{a})$ Determine the gradient of $\boldsymbol{f}$ at the point $(\mathbf{1},\mathbf{2})$.

The directional derivative of function f at some point (x_0,y_0) in the direction of a vector $\vec{v}$ is defined as:

$$\partial_{\vec{v}}f=\langle\nabla f,\vec{v}\rangle=v_1\,f_x(x_0,y_0)+v_2\,f_y(x_0,y_0)$$

Therefore, we can write the following system of equations:

$$\begin{cases}3f_x(1,2)+4f_y(1,2)=2 & (\alpha)\\ 5f_x(1,2)+12f_y(1,2)=1 & (\beta)\end{cases}$$

Now, by $(\beta)-3(\alpha)$:

$$5f_x(1,2)+12f_y(1,2)-9f_x(1,2)-12f_y(1,2)=1-6\ \Rightarrow\ -4f_x(1,2)=-5\ \Rightarrow\ f_x(1,2)=\frac{5}{4}$$

and by $5(\alpha)-3(\beta)$:

$$15f_x(1,2)+20f_y(1,2)-15f_x(1,2)-36f_y(1,2)=10-3\ \Rightarrow\ -16f_y(1,2)=7\ \Rightarrow\ f_y(1,2)=-\frac{7}{16}$$

Therefore, the gradient of the function f at the point $(1,2)$ will be:

$$\nabla f(1,2)=\left(\frac{5}{4},-\frac{7}{16}\right)$$

Another way of interpreting this exercise would be to consider the normalised vectors. In this case, we would have the following system of equations:

$$\begin{cases}{}^3/_5\ f_x(1,2)+{}^4/_5\ f_y(1,2)=2\\ {}^5/_{13}\ f_x(1,2)+{}^{12}/_{13}\ f_y(1,2)=1\end{cases}\ \Rightarrow\ \begin{cases}3f_x(1,2)+4f_y(1,2)=10 & (\alpha)\\ 5f_x(1,2)+12f_y(1,2)=13 & (\beta)\end{cases}$$

Now, by $(\beta)-3(\alpha)$:

$$5f_x(1,2)+12f_y(1,2)-9f_x(1,2)-12f_y(1,2)=13-30\ \Rightarrow\ -4f_x(1,2)=-17\ \Rightarrow\ f_x(1,2)=\frac{17}{4}$$

and by $5(\alpha)-3(\beta)$:

$$15f_x(1,2)+20f_y(1,2)-15f_x(1,2)-36f_y(1,2)=50-39\ \Rightarrow\ -16f_y(1,2)=11\ \Rightarrow\ f_y(1,2)=-\frac{11}{16}$$

Therefore, the gradient of the function f at the point $(1,2)$ will be:

$$\nabla f(1,2) = \left(\frac{17}{4}, -\frac{11}{16}\right)$$

Note that both interpretations of the exercise are totally valid!

$(\boldsymbol{b})$ What is the maximum rate of change of the function $\boldsymbol{f}$ at the point $(\mathbf{1},\mathbf{2})$? And in what direction does this maximum occur?

Note that as in the previous exercise, we can have the two different interpretations. We will now solve it considering the non-normalised vectors, but the other approach is as valid!

The maximum rate of change of the function f at the point $(1,2)$ corresponds to the greatest directional derivative at such point. The directional derivative of a function in the direction of some vector $\vec{v}$ can be calculated as:

$$\partial_{\vec{v}} f = \langle \nabla f, \vec{v} \rangle = \|\nabla f\| \|\vec{v}\| \cos\alpha$$

Since $\vec{v}$ is a unit vector, that is $\|\vec{v}\| = 1$, and the maximum value for $\cos\alpha$ is 1, the maximum rate of change will correspond to the norm of the gradient:

$$\|\nabla f\| = \sqrt{\left({}^{5}/_{4}\right)^2 + \left(-{}^{7}/_{16}\right)^2} = \sqrt{\frac{5^2 * 16 + 7^2}{256}} = \sqrt{\frac{449}{256}} = \frac{\sqrt{449}}{16}$$

Since the greatest directional derivative is:

$$\partial_{\vec{v}} f = \nabla f \cdot \vec{v} = \|\nabla f\|$$

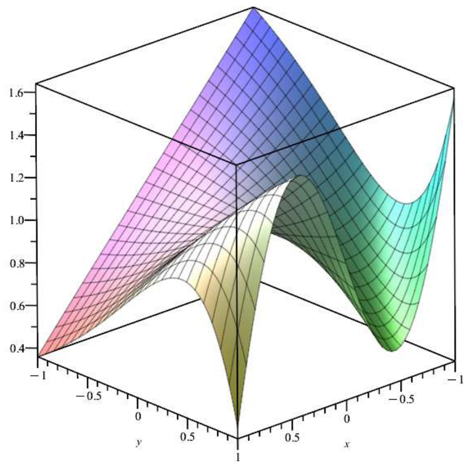
we can easily determine the direction $\vec{v}$ as:

$$v = \frac{\nabla f(1,2)}{\|\nabla f(1,2)\|} = \frac{16}{\sqrt{449}} \left(\frac{5}{4}, -\frac{7}{16}\right) = \frac{1}{\sqrt{449}} (20, -7)$$

EXERCISE 2: Consider the function $\boldsymbol{f(x,y)}$ defined by:

$$\boldsymbol{f(x,y) = \sin(xe^y) - x + 3}$$

$(\boldsymbol{a})$ Compute all the first and second order partial derivatives.



$$\frac{\partial}{\partial x} f(x,y) = \cos(xe^y) * e^y - 1$$

$$\frac{\partial}{\partial y} f(x,y) = \cos(xe^y) * xe^y$$

$$\frac{\partial^2}{\partial x^2} f(x,y) = -\sin(xe^y) * e^{2y}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = -\sin(xe^y) * x^2 e^{2y} + \cos(xe^y) * xe^y$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = -\sin(xe^y) * xe^{2y} + \cos(xe^y) * e^y$$

$$\frac{\partial^2}{\partial x \partial y} f(x,y) == -\sin(xe^y) * xe^{2y} + \cos(xe^y) * e^y$$

As the function is twice differentiable and its derivatives are continuous, its second order mixed partial derivatives are equal.

(b) A given fact is that $\boldsymbol{f(0.2, 0.1) = 3.01924}$ accurate to $\mathbf{5}$ decimal places. Taking $\mathbf{(0, 0)}$ as the reference point, use linear approximation to find an approximation of $\boldsymbol{f(0.2, 0.1)}$.

The linear approximation of the function at the point $(0, 0)$ is given by:

$$T_1(x, y) = f(0,0) + \nabla f(0,0) \cdot (x,y) = f(x,y) + x\, f_x(0,0) + y\, f_y(0,0)$$

The gradient of f at the point $(0,0)$ can be calculated by substituting in the derivatives we calculated in (a).

$$\nabla f(0,0) = \left(f_x(0,0), f_y(0,0)\right) = (0,0)$$

and therefore:

$$T_1(0.2, 0.1) = f(0,0) + (0,0) \cdot (0.2, 0.1) = f(0,0) = 3$$

(c) By referring to the idea of a tangent plane, explain why you obtained the answer you did in part $\boldsymbol{(b)}$.

Since the gradient of the function at $(0,0)$ is the (0,0) vector, the tangent plane will be horizontal, so the linear approximation corresponds to the value of the function at that point.

(d) Use a $\mathbf{2^{nd}}$ order Taylor polynomial to find an approximation of $\boldsymbol{f(0.2, 0.1)}$. Is this approximation better or worse than the linear approximation?

The second order Taylor polynomial centred at $(0,0)$ can be calculated as:

$$T_2(x,y) = T_1(x,y) + \frac{1}{2}[x \quad y]\, H(0,0) \begin{bmatrix} x \\ y \end{bmatrix} = T_1(x,y) + \frac{1}{2}[x \quad y] \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let's first calculate the second order partial derivatives at the point $(0,0)$.

$$f_{xx}(0,0) = f_{yy}(0,0) = 0 \quad \text{and} \quad f_{xy}(0,0) = 1$$

Therefore, the new approximation for the function at the point $(0,0)$ gives us:

$$T_2(x,y) = T_1(x,y) + \frac{1}{2}[x \quad y] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T_1(x,y) + \frac{1}{2}[x \quad y] \begin{bmatrix} y \\ x \end{bmatrix} = T_1(x,y) + \frac{2xy}{2} = T_1(x,y) + xy$$

Substituting with the point $(0.2, 0.1)$:

$$T_2(0.2, 0.1) == T_1(0.2, 0.1) + 0.2 * 0.1 = 3 + 0.02 = 3.02$$

We can clearly see by comparing both approximations that the second order Taylor polynomial gives a better approximation than the linear one.

(e) Find a critical point of $f(x,y)$ and determine if it is a local minimum, local maximum or saddle point. You do not need to find all of them.

The critical points of the function $f(x,y)$ are those where the gradient is $\mathbf{0}$. In part (c), we saw that:

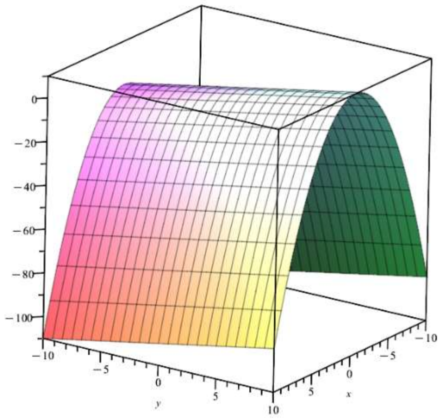
$$\nabla f(0,0) = (0,0)$$

which means that the point $(0,0)$ is a critical point of the function f. We can determine if $(0,0)$ is a maximum, minimum or saddle point by calculating the determinant of the Hessian matrix $H(0,0)$:

$$\det(H(0,0)) = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0) = 0 * 0 - 1^2 = -1$$

Because $\det(H(0,0)) < 0$, the point $(0,0)$ is a saddle point.

EXERCISE 3: Let $\boldsymbol{f(x,y) = y - x^2}$.



(a) Make a sketch of the level curves $\boldsymbol{f(x,y) = 1}$ and $\boldsymbol{f(x,y) = 2}$ on the same axes, that is, make a contour plot with these two level curves.

We can draw the contour plot with Maple:

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with(plots)
contourplot(y − x², contours = [1, 2])
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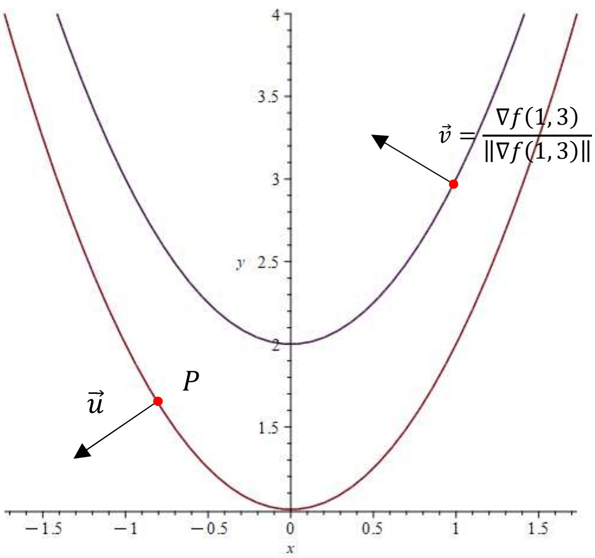

Note that the red curve corresponds to the level curve $f = 1$ and the purple one to $f = 2$.

(b) On the above plot, draw a point $\boldsymbol{P}$ and a vector $\vec{\boldsymbol{u}}$ (with initial point at $\boldsymbol{P}$) such that the directional derivative $\boldsymbol{D_{\vec{u}}(P)}$ is negative.

Since the gradient vector points to the direction of maximum increase and is orthogonal to the level curve, we can consider any vector $\vec{u}$ pointing away from the level curve given by $f(x,y) = 2$.

$(\boldsymbol{c})$ At the point $(\mathbf{1}, \mathbf{3})$, find the direction in which $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ is increasing the fastest. Sketch the point and the vector on the above plot.

The directional derivative of a function in the direction of some vector $\vec{v}$ can be calculated as:

$$\partial_{\vec{v}} f = \langle \nabla f, \vec{v} \rangle = \|\nabla f\| \|\vec{v}\| \cos \alpha$$

Since $\vec{v}$ is a unit vector, that is $\|\vec{v}\| = 1$, and the maximum value for $\cos \alpha$ is 1, the maximum directional derivative is given by:

$$\partial_{\vec{v}} f = \nabla f \cdot \vec{v} = \|\nabla f\|$$

Therefore, the direction $\vec{v}$ in which f is increasing the fastest will be:

$$\vec{v} = \frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = \frac{1}{\sqrt{(-2)^2 + 1^2}} (-2, 1) = \frac{1}{\sqrt{5}} (-2, 1) = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$