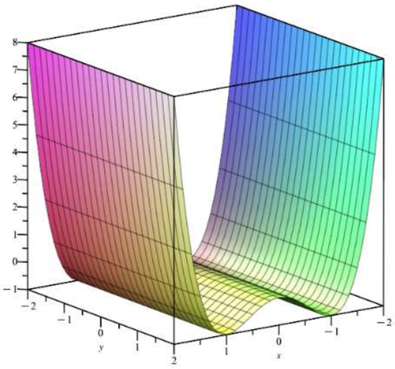
WEEK 4 – CLASS EXERCISES

EXERCISE 1: A flat circular plate has the shape of the region $\boldsymbol{x^2 + y^2 \leq 1}$. The plate (including the boundary $\boldsymbol{x^2 + y^2 = 1}$) is heated so that the temperature $\boldsymbol{T(x, y)}$ at any point $\boldsymbol{(x, y)}$ is given by $\boldsymbol{T(x, y) = x^4 - 4x^2 + 2y^2}$. Locate the hottest and coldest points of the plate and determine the temperature at each of these points.

<u>Interior:</u> Let's start by finding the critical points of the function in the interior, that is, in the region $x^2 + y^2 < 1$. The critical points of the function $T(x, y)$ are those in which the gradient of T is the $(0, 0)$ vector. That is:



$$\nabla T = 0 \quad \Longrightarrow \quad (4x^3 - 8x, 4y) = (0, 0) \quad \Longrightarrow \quad \begin{cases} 4x\,(x^2 - 2) = 0 \\ 4y = 0 \end{cases}$$

Therefore, the critical points inside the plate are attained when $y = 0$ and either $x = 0$ or $x = \pm\sqrt{2}$. Since we have the constraint $x^2 + y^2 < 1$, the only possible extremum inside the plate is $(0,0)$, with a temperature of $T(0,0) = 0$.

<u>Boundary:</u> In order to study the possible extrema in the boundary let's use the method of Lagrange multipliers, where our constraint is $g(x, y) = 0$, where $g(x) = x^2 + y^2 - 1$. Let's look for the points where:

$$\nabla T = \lambda \nabla g \quad \Longrightarrow \quad (4x^3 - 8x, 4y) = \lambda\,(2x, 2y) \quad \Longrightarrow \quad \begin{cases} 4x\,(x^2 - 2) = 2\lambda x \\ 4y = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \quad \Longrightarrow \quad \begin{cases} 2x\,(2x^2 - 4 - \lambda) = 0 & (1) \\ 2y\,(2 - \lambda) = 0 & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

From equation (2), we know that either $y = 0$ or $\lambda = 2$. In the case $y = 0$, from equation (3) we would get $x = \pm 1$. In the case of $\lambda = 2$, from equation (1) we get that either $x = 0$ and then by (3) $y = \pm 1$, or $x = \pm\sqrt{3}$, but the latter does not lie on the boundary and will never satisfy the third equation. Therefore, the possible extrema in the boundary are the points $(0, \pm 1)$, with a temperature $T(0, \pm 1) = 2$, and the points $(\pm 1, 0)$, with a temperature $T(\pm 1, 0) = -3$.

By comparing the values at each of the points that we obtained in both the interior and the boundary, we can finally state that the coldest points in the plate will be $(\pm 1, 0)$ with a temperature of -3, and the hottest ones will be $(0, \pm 1)$, with a temperature of 2.

> We could have also studied the boundary of the function by reducing it to one variable. We would then have:
>
> $$T(x, y) = x^4 - 4x^2 + 2(1 - x^2) = x^4 - 4x^2 + 2 - 2x^2 = x^4 - 6x^2 + 2 = s^2 - 6s + 2 \quad \text{where we let } s = x^2$$
>
> Since $s = x^2$ can only be between 0 and 1 so that the point is inside the plate, we then have to study the critical points of the function $f(s) = s^2 - 6s + 2$ in the interval $s \in [0, 1]$. However, we have that $f'(s) = 2s - 6 = 0$ if and only if $s = 3$, which is outside of the allowed range for s. This implies that the function has no critical points in $(0,1)$, so we are only left with the boundaries: when $s = 1$, we then have the points $(\pm 1, 0)$, and when $s = 0$, we have $(0, \pm 1)$. We can therefore calculate its temperatures and conclude the problem as we did before.

EXERCISE 2: Use the method of Lagrange multipliers to find the absolute extrema of the function $\boldsymbol{f(x,y) = 4xy}$ on the region $\boldsymbol{4x^2 + y^2 \leq 8}$.

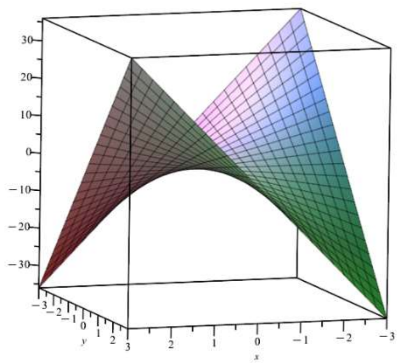
<u>Interior:</u> The extrema of the function $f(x,y)$ are those in which:

$$\nabla f = 0 \quad \Longrightarrow \quad (4y, 4x) = (0,0) \quad \Longrightarrow \quad \begin{cases} 4y = 0 \\ 4x = 0 \end{cases}$$

The only solution is the point $(0,0)$, where we have $f(0,0) = 0$.

<u>Boundary:</u> We can use Lagrange multipliers by applying the constraint $g(x,y) = 0$, where $g(x) = 4x^2 + y^2 - 8$:

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad (4y, 4x) = \lambda\,(8x, 2y) \quad \Longrightarrow \begin{cases} 4y = 8\lambda x \\ 4x = 2\lambda y \\ 4x^2 + y^2 = 8 \end{cases} \Longrightarrow \begin{cases} y = 2\lambda x & (1) \\ 2x = \lambda y & (2) \\ 4x^2 + y^2 = 8 & (3) \end{cases}$$



By combining equations (1) and (2), we get that $y = \lambda^2 y$, which implies that either $y = 0$, which leads to the point $(0,0)$, or $\lambda = 1$. Since the point $(0,0)$ does not satisfy equation (3), or in other words, it is not on the boundary, we are only left with the case $\lambda^2 = 1$, that is, $\lambda = \pm 1$. This leads to $y = \pm 2x$, which, by substituting in equation (3), we get the points $(\pm 1, \pm 2)$.

By calculating evaluating the function f at those four points and comparing their values to the point $(0,0)$ which we obtained previously, we can conclude that the absolute maxima are $(1,2)$ and $(-1,-2)$, with a value of $f(1,2) = f(-1,-2) = 8$, and the absolute minima are $(1,-2)$ and $(-1,2)$, with a value of $f(1,-2) = f(-1,2) = -8$.

EXERCISE 3: Let $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, ..., $\boldsymbol{a_n}$ be positive numbers:

$(\boldsymbol{a})$ Find the maximum value of the expression $\boldsymbol{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$ if the variables $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, ..., $\boldsymbol{x_n}$ are restricted so that the sum of their squares is one.

We can use Lagrange multipliers with the following constraint:

$$g(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1 = \sum_{i=1}^{n} x_i^2 - 1 = 0 \quad \Longrightarrow \quad \sum_{i=1}^{n} x_i^2 = 1$$

Let us define $f(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum a_i x_i$. We then want:

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad (a_1, a_2, \dots, a_n) = \lambda\,(2x_1, 2x_2, \dots, 2x_n) \quad \Longrightarrow \quad \begin{cases} 2\lambda x_i = a_i \quad \forall i \in \mathbb{N} \text{ and } i \leq n \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{cases}$$

By rearranging the first equation, we get $x_i = \frac{a_i}{2\lambda}$, and by substituting in the second equation:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \left(\frac{a_n}{2\lambda}\right)^2 + \left(\frac{a_n}{2\lambda}\right)^2 + \cdots + \left(\frac{a_n}{2\lambda}\right)^2 = \frac{1}{4\lambda^2}\sum_{i=1}^{n} a_i^2 = 1$$

which then means that:

$$\frac{1}{4\lambda^2}\sum_{i=1}^{n} a_i^2 = 1 \quad\Longrightarrow\quad \lambda^2 = \frac{1}{4}\sum_{i=1}^{n} a_i^2 \quad\Longrightarrow\quad \lambda = \pm\frac{1}{2}\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

We get the following expression for x_i:

$$x_i = \frac{a_i}{2\lambda} = x_i = \pm\, a_i \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2}$$

Note that, since the coefficients $\boldsymbol{a_i}$ are positive, we have found two points which are symmetric with respect to the origin. We now just need to evaluate the function at those two points. Let us denote the point with positive coefficients as $(+)$ and the one with negative coefficients as $(-)$. Then:

$$f(+) = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i^2 \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2} = \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2} \sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

$$f(-) = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} -a_i^2 \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2} = -\left(\sum_{i=1}^{n} a_i^2\right)^{-1/2} \sum_{i=1}^{n} a_i^2 = -\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

We can therefore conclude that the maximum of the function is obtained at the point with coordinates

$$x_i = a_i \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2},$$

where the function has a value of

$$f(+) = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

(b) What is the minimum value of $\boldsymbol{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n}$ in this case?

We can directly see from part (a) that the minimum will be attained at the point with coordinates

$$x_i = -\, a_i \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2},$$

where the function has a value of

$$f(-) = -\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

(c) Give an interpretation of your results using the dot product in the case $\boldsymbol{n = 2}$.

Our function is now $f(x_1, x_2) = a_1 x + a_2 x_2$, which can be defined as an inner product between the vectors $\vec{a}$ and $\vec{x}$:

$$f(\vec{x}) = \vec{a} \cdot \vec{x} = ||\vec{a}|| ||\vec{x}|| \cos \alpha$$

The restriction of the sum of the squares of the components of x being 1 implies that x is a unit vector, that is, $||\vec{x}|| = 1$. Since also $||\vec{a}||$ is fixed and determined by the coefficients of the function, the value of our function relies on the cosine of the angle between both vectors. Therefore, the absolute maximum is attained when $\alpha = 0$, that is, both vectors point in the same direction, and, similarly, the absolute minimum occurs when they point in opposite directions. We can therefore calculate x as follows:

$$\vec{a} \cdot \vec{x} = \pm \, ||\vec{a}|| \quad \Longrightarrow \quad \vec{x} = \pm \frac{\vec{a}}{||\vec{a}||} = \pm \frac{(a_1, a_2)}{\sqrt{a_1^2 + a_2^2}}$$

REMARK: Recall that the directional derivative is the dot product of the gradient vector and the corresponding vector. Under the constraint of the vector being an unit vector, notice that this interpretation is the same as the one for proving that the gradient vector is the direction of maximum increase of a function.

EXERCISE 4: Find approximate solutions of the following system of equations:

$$\begin{cases} \boldsymbol{x^2 + y^2 - 1 = 0} \\ \boldsymbol{y - e^x = 0} \end{cases}$$

For that, we can program the Newton's Method on a computer (for example, using Maple, Python, MatLab) to generate the required approximations. In the exercise class you could try to program this or just discuss how you would implement practically this using Newton's method.

Let us first remember how Newton's method works:

We start by defining a matrix F containing both functions $f(x, y) = x^2 + y^2 - 1$ and $g(x, y) = y - e^x$ and calculating its Jacobian matrix J and its inverse J^{-1}. We define our starting point $P_0(x_0, y_0)$ and approximate the solution by iterating as many times as we want with the next formula:

$$P_i = P_{i-1} - J^{-1}(P_{i-1}) * F(P_{i-1})$$

Note that the system of equations above has two solutions. However, by applying this method to some initial point P_0, we will only reach one of them. We need to try with different initial points in order to be able to get all the solutions. Below you can see the code in Maple, where we have chosen $P_0(2, 2)$ as a starting point and we get the approximate solution $(9.95 \cdot 10^{-1} \ , 1.0000)$, while the exact solution is $(0, 1)$.

```
with(LinearAlgebra); with(VectorCalculus); with(plots)
f := x^2 + y^2 - 1
```

$$f := x^2 + y^2 - 1$$

```
g := y - e^x
```

$$g := y - e^x$$

```
F := Matrix(2, 1, [f, g])
```

$$F := \begin{bmatrix} x^2 + y^2 - 1 \\ y - e^x \end{bmatrix}$$

```
J := Jacobian([f, g], [x, y])
```

$$J := \begin{bmatrix} 2x & 2y \\ -e^x & 1 \end{bmatrix}$$

```
Jinv := MatrixInverse(J)
```

$$Jinv := \begin{bmatrix} \frac{1}{2\,(y\,e^x + x)} & -\frac{y}{y\,e^x + x} \\ \frac{e^x}{2\,(y\,e^x + x)} & \frac{x}{y\,e^x + x} \end{bmatrix}$$

```
x[0] := 2; y[0] := 2
```

$$x_0 := 2$$

$$y_0 := 2$$

```
for i from 0 to 6 do
A := Matrix(2, 1, [x[i], y[i]]) - subs(x = x[i], y = y[i], Jinv) . subs(x = x[i], y = y[i], F);
x[i + 1] := evalf(A[1, 1]);
y[i + 1] := evalf(A[2, 1]);
end
```

$$A := \begin{bmatrix} 2 - \frac{7}{2\,(2\,e^2 + 2)} + \frac{2\,(2 - e^2)}{2\,e^2 + 2} \\ 2 - \frac{7\,e^2}{2\,(2\,e^2 + 2)} - \frac{2\,(2 - e^2)}{2\,e^2 + 2} \end{bmatrix}$$

$$x_1 := 1.149003652$$

$$y_1 := 1.100996348$$

$$A := \begin{bmatrix} 0.494039131500000 \\ 1.08860343750000 \end{bmatrix}$$

$$x_2 := 0.494039131500000$$

$$y_2 := 1.08860343750000$$

$$A := \begin{bmatrix} 0.136891358936767 \\ 1.05358510381892 \end{bmatrix}$$

$$x_3 := 0.136891358936767$$

$$y_3 := 1.05358510381892$$

$$A := \begin{bmatrix} 0.0160782309758589 \\ 1.00816671840998 \end{bmatrix}$$

$$x_4 := 0.0160782309758589$$

$$y_4 := 1.00816671840998$$

$$A := \begin{bmatrix} 0.000282838819896544 \\ 1.00015677451922 \end{bmatrix}$$

$$x_5 := 0.000282838819896544$$

$$y_5 := 1.00015677451922$$

$$A := \begin{bmatrix} 9.22340954987756 \times 10^{-8} \\ 1.00000005225374 \end{bmatrix}$$

$$x_6 := 9.22340954987756 \times 10^{-8}$$

$$y_6 := 1.00000005225374$$

$$A := \begin{bmatrix} 9.94568958071619 \times 10^{-15} \\ 1.00000000000001 \end{bmatrix}$$

$$x_7 := 9.94568958071619 \times 10^{-15}$$

$$y_7 := 1.00000000000001$$

Similarly, here's the code in MatLab, where we have now used $P_0(-2, 2)$ as an initial point. We now get the approximate solution $(-0.916, 0.339)$, and the error with the exact solution is less than $0.005\ \%$.

```
clearvars
close all
clc

syms x y f(x,y) g(x,y)
f(x,y) = x^2 + y^2 - 1;
g(x,y) = y - exp(x);
F = [f(x,y); g(x,y)];
J = jacobian(F,[x,y]);
Jinv = inv(J);
disp("Let's use Newton's method to solve the following system of equations:")
disp('Let us define f us f(x,y) = ' + string(f) + ' and g as g(x,y) = ' + string(g));
disp(' ')

x_0 = -2;
y_0 = 2;
loops = 5;
disp(['Starting from the point (',num2str(x_0),',',num2str(y_0),') and iterating ',num2str(loops),' times:'])
for i = 1:loops
    A = [x_0;y_0] - subs(Jinv,[x,y],[x_0,y_0]) * subs(F,[x,y],[x_0,y_0]);
    new = subs(A,[x,y],[x_0,y_0]);
    x_0 = double(new(1,1));
    y_0 = double(new(2,1));
    disp(['Iteration ',num2str(i),': We now have x = ',num2str(x_0,8),' and y = ',num2str(y_0,8)])
end

[solx,soly] = vpasolve([f,g],[-2,2]);
errorx = abs(solx - x_0) * 100 / x_0;
errory = abs(soly - y_0) * 100 / y_0;
maxerror = double(max(errorx, errory));
disp(' ')
disp(['With a maximum relative error of  ',num2str(maxerror,6),' %, we can conclude that one solution to'])
disp(['the system of equations is the point (',num2str(x_0,8),',',num2str(y_0,8),')'])
```

```
MATLAB Command Window                                                    Page 1

Let's use Newton's method to solve the following system of equations:
Let us define f us f(x,y) = x^2 + y^2 - 1 and g as g(x,y) = y - exp(x)

Starting from the point (-2,2) and iterating 5 times:
Iteration 1: We now have x = -2.1326118 and y = 0.11738823
Iteration 2: We now have x = -1.2919822 and y = 0.21816489
Iteration 3: We now have x = -0.99106298 and y = 0.35739592
Iteration 4: We now have x = -0.9212871 and y = 0.39708147
Iteration 5: We now have x = -0.91658469 and y = 0.39987803

With a maximum relative error of  0.00331314 %, we can conclude that one solution to
the system of equations is the point (-0.91658469,0.39987803)
```

We can now plot the intersection of the two curves with the plane $z = 0$. Indeed, we can clearly see that the approximations that we calculated before are very close to the exact solutions of our system of equations, which correspond to the points of intersection of the two curves.

$\mathrm{f} := x^2 + y^2 - 1$

$$f := x^2 + y^2 - 1 \qquad (1)$$

$\mathrm{g} := \mathrm{y} - \mathrm{e}^x$

$$g := y - e^x \qquad (2)$$

$inter := implicitplot([f = 0, g = 0], x = -2..2, y = -2..2, color = [green, blue], thickness = 3)$