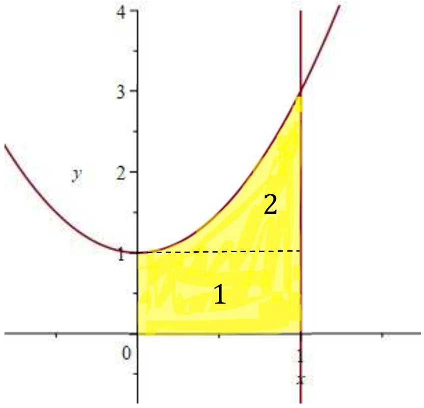
WEEK 5 – CLASS EXERCISES

EXERCISE 1: Reverse the order for integration for

$$\int_0^1 \int_0^{2x^2+1} \boldsymbol{f(x,y)\,dy\,dx}$$

that is, write as an integral of the form $\int\int \ldots\ \boldsymbol{dx\,dy}$.



In order to make it easier, we can plot the area of integration using Maple:

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with(plots) :
a := plot(2x^2 + 1, x = -2 ..2, y = -4 ..4) :
b := implicitplot(x = 1, x = -2 ..2, y = -4 ..4) :
display(a, b)
```

Let's analyse the two parts of the graph separately. Part 1 is very straightforward: y goes from 0 to 1 and so does x:

$$(1)\quad \int_0^1 \int_0^1 f(x,y)\,dx\,dy$$

For part 2, we can use the picture to realise that we will now consider values of y that go from 1 to 3. However, in this case, the x coordinate starts from the parabola and goes until 1. We can find the function of the parabola in terms of y as follows:

$$y = 2x^2 + 1 \quad \Rightarrow \quad x^2 = \frac{y-1}{2} \quad \Rightarrow \quad x = \pm\sqrt{\frac{y-1}{2}}$$

Since x is always positive, we will consider only the positive square root. That is:

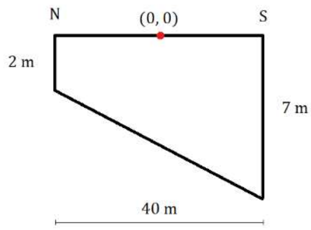
$$(2)\quad \int_1^3 \int_{\sqrt{\frac{y-1}{2}}}^1 f(x,y)\,dx\,dy$$

We therefore have the following:

$$\int_0^1 \int_0^{2x^2+1} f(x,y)\,dy\,dx = \int_0^1 \int_0^1 f(x,y)\,dx\,dy + \int_1^3 \int_{\sqrt{\frac{y-1}{2}}}^1 f(x,y)\,dx\,dy$$

EXERCISE 2: A swimming pool is circular with a $\mathbf{40}$ metre diameter. The depth is constant along east-west lines and increases linearly from $\mathbf{2}$ metres at the south end to $\mathbf{7}$ metres at the north end. Find the volume of the pool.

Since the depth is constant along the east-west lines, let's make a drawing of the pool on the lateral side from north to south. Our pool would look something like the following:



If we centre the swimming pool in the origin, we will then be able to use polar coordinates (r, θ) in order to solve such problem. The depth $D(y)$ of the pool can be then represented as a straight line between the points $(-20, -2)$ and $(20, -7)$. The equation of the line will be:

$$D(y) = \frac{7-2}{20+20}(y+20) + 2 = \frac{5}{40}(y+20) + 2 = \frac{5y}{40} + \frac{5}{2} + 2 = \frac{5y}{40} + \frac{9}{2}$$

which can be rewritten in polar coordinates as follows:

$$D(r, \theta) = \frac{5}{40} r \sin\theta + \frac{9}{2}$$

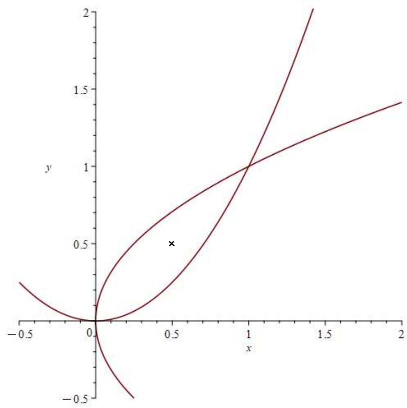
Therefore, the volume of the swimming pool will be:

$$\int_0^{20}\int_0^{2\pi} D(r,\theta)\, r\, d\theta\, dr = \int_0^{20}\int_0^{2\pi}\left(\frac{5}{40} r \sin\theta + \frac{9}{2}\right) r\, d\theta\, dr = \int_0^{20}\left[-\frac{5}{40} r\cos\theta + \frac{9}{2}\theta\right]_{\theta=0}^{\theta=2\pi} r\, dr$$

$$\int_0^{20} 9\pi r\, dr = \left[\frac{9\pi r^2}{2}\right]_0^{20} = \frac{9\pi * 20^2}{2} = 1800\pi \approx 5654.9\, m^3$$

EXERCISE 3: Find the centre of mass of a two-dimensional plate that occupies the region enclosed by the parabolas $\boldsymbol{x = y^2}$ and $\boldsymbol{y = x^2}$, and has density function $\boldsymbol{\rho(x, y) = \sqrt{x}}$.

It can be seen from the plot on the right that the integration for x will be from 0 to 1, and the integration for y will go form the x^2 to $\sqrt{x}$. We can start by calculating the total mass of the plate:



$$M = \int_0^1\int_{x^2}^{\sqrt{x}} \rho(x,y)\, dy\, dx = \int_0^1\int_{x^2}^{\sqrt{x}} \sqrt{x}\, dy\, dx =$$

$$\int_0^1 \left[y\sqrt{x}\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 x - x^2\sqrt{x}\, dx = \left[\frac{x^2}{2} - \frac{2x^{7/2}}{7}\right]_0^1 = \frac{1}{2} - \frac{2}{7} = \frac{3}{14}$$

Also, the moments of the x coordinate will be:

$$M_x = \int_0^1\int_{x^2}^{\sqrt{x}} y\,\rho(x,y)\, dy\, dx = \int_0^1\int_{x^2}^{\sqrt{x}} y\sqrt{x}\, dy\, dx = \int_0^1\left[\frac{y^2}{2}\sqrt{x}\right]_{y=x^2}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 x\sqrt{x} - x^4\sqrt{x}\, dx =$$

$$= \frac{1}{2}\int_0^1 x^{3/2} - x^{9/2}\, dx = \frac{1}{2}\left[\frac{2x^{5/2}}{5} - \frac{2x^{11/2}}{11}\right]_0^1 = \frac{1}{5} - \frac{1}{11} = \frac{6}{55}$$

Similarly for y:

$$M_y = \int_0^1\int_{x^2}^{\sqrt{x}} x\,\rho(x,y)\, dy\, dx = \int_0^1\int_{x^2}^{\sqrt{x}} x\sqrt{x}\, dy\, dx = \int_0^1\left[yx^{3/2}\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 x^2 - x^{7/2}\, dx =$$

$$= \left[\frac{x^3}{3} - \frac{2x^{9/2}}{9}\right]_0^1 = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$$

Therefore, the coordinates of the centre of mass will be:

$$c = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{1}{9} * \frac{14}{3}, \frac{6}{55} * \frac{14}{3}\right) = \left(\frac{14}{27}, \frac{28}{55}\right)$$

EXERCISE 4: Calculus: Early Trascendentals – David Guichard and friends. Section 15.1.

- Exercise 15.1.1:

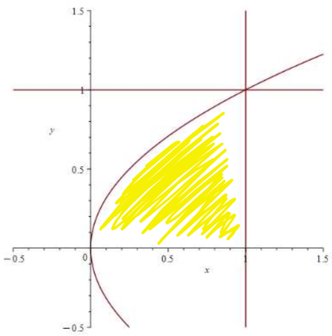
$$\int_0^2 \int_0^4 1 + x\, dy\, dx = \int_0^2 [\, y\,(1+x)\,]_{y=0}^{y=4}\, dx = 4\int_0^2 1 + x\, dx = 4\left[x + \frac{x^2}{2}\right]_0^2 = 4\left(2 + \frac{2^2}{2}\right) = 16$$

- Exercise 15.1.3:

$$\int_1^2 \int_0^y xy\, dx\, dy = \int_0^2 \left[\frac{x^2}{2}\, y\right]_{x=0}^{x=y} dx = \frac{1}{2}\int_1^2 y^3\, dy = \frac{1}{2}\left[\frac{y^4}{4}\right]_0^2 = \frac{1}{2}\left(\frac{2^4}{4} - \frac{1}{4}\right) = \frac{15}{8}$$

- Exercise 15.1.10:

Because the given integral is impossible to solve like that, we will need to change the order of integration. Let's plot the area of integration using Maple in order to understand it more easily. From the plot on the right, we can easily see that x will go from 0 to 1 and the values of y will go from 0 to $\sqrt{x}$. Therefore:

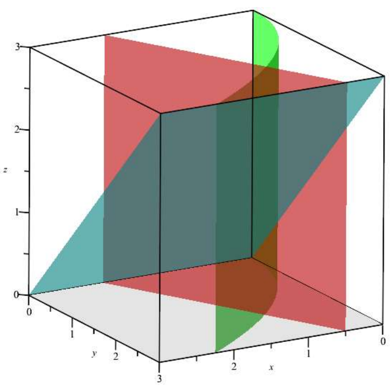


$$\int_0^1 \int_{y^2}^1 y \sin x^2\, dx\, dy = \int_0^1 \int_0^{\sqrt{x}} y \sin x^2\, dy\, dx$$

$$\int_0^1 \left[\frac{y^2}{2} \sin x^2\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{2} \sin x^2\; dx = -\left[\frac{1}{4}\cos x^2\right]_0^1 = -\frac{\cos(1) - 1}{4} = \frac{1 - \cos(1)}{4}$$

- Exercise 15.1.20: Find the volume in the first octant bounded by $\boldsymbol{y^2 = 4x}$, $\boldsymbol{2x + y = 4}$, $\boldsymbol{z = y}$ and $\boldsymbol{z = 0}$.

Let's start by plotting the volume. With the help of the plot and the following equations, let's try to find the bounds for the coordinates x, y and z. Note that, because we are in the first quadrant, the values for x, y and z will be non-negative.



$$\begin{cases} y^2 = 4x & (1) \\ 2x + y = 4 & (2) \\ z = y & (3) \\ z = 0 & (4) \end{cases}$$

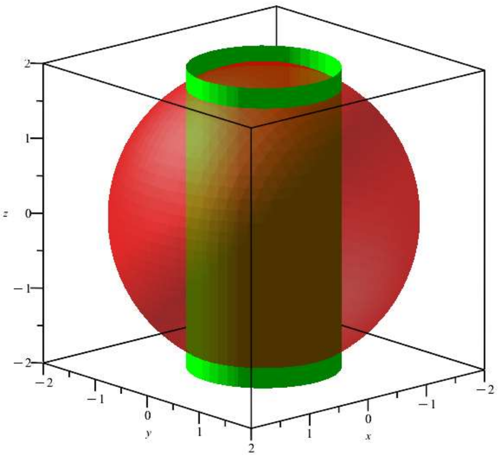
The intersection point in the xy plane between the green (1) and red (2) surfaces can be determined with a system of equations between the curves. Such intersection point is $(1, 2)$. Therefore, while the values of x go from the green curve, that is, $x = {y^2}/{4}$, until those determined by the red plane, that is, $x = 2 - {y}/{2}$, the values of y simply go from 0 to 2 (we got this upper bound from the intersection point). Since the height is determined by the function $z = y$, the volume will be:

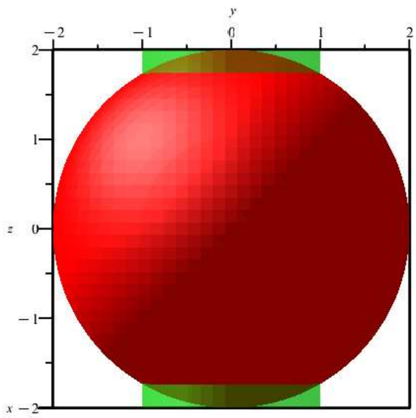
$$\int_0^2 \int_{\frac{y^2}{4}}^{2-\frac{y}{2}} y \, dx \, dy = \int_0^2 [\, yx \,]_{x=\frac{y^2}{4}}^{x=2-\frac{y}{2}} dy = \int_0^2 2y - \frac{y^2}{2} - \frac{y^3}{4} \, dy =$$

$$= \left[y^2 - \frac{y^3}{6} - \frac{y^4}{16} \right]_0^2 = 2^2 - \frac{2^3}{6} - \frac{2^4}{16} = 4 - \frac{8}{6} - 1 = 3 - \frac{4}{3} = \frac{5}{3}$$

EXERCISE 5: Calculus: Early Trascendentals – David Guichard and friends. Section 15.2.

- Exercise 15.2.2: Find the volume inside both $\boldsymbol{r = 1}$ and $\boldsymbol{r^2 + z^2 = 4}$.



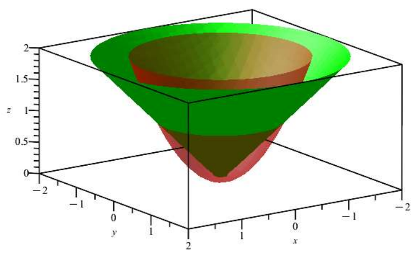


Note that the volume is symmetric with respect to the plane $z = 0$, so we will calculate the total volume as twice the volume in the upper half-space. We can determine the total height for z by solving the equation of the sphere. We obtain $z = \pm\sqrt{4 - r^2}$, and because we will first consider the upper-half space, we will consider the positive z. The volume of the whole volume will be:

$$2 \int_0^{2\pi} \int_0^1 \sqrt{4 - r^2} \, r \, dr \, d\theta = 2 \int_0^{2\pi} \left[-\frac{1}{3} (4 - r^2)^{3/2} \right]_{r=0}^{r=1} d\theta = -\frac{2}{3} \int_0^{2\pi} 3^{3/2} - 4^{3/2} \, d\theta =$$

$$= -\frac{4\pi}{3} \left(3^{3/2} - 2^3 \right) = \frac{4\pi}{3} \left(8 - 3\sqrt{3} \right) u^3$$

- Exercise 15.2.6: Find the volume between $\boldsymbol{x^2 + y^2 = z^2}$ and $\boldsymbol{x^2 + y^2 = z}$.

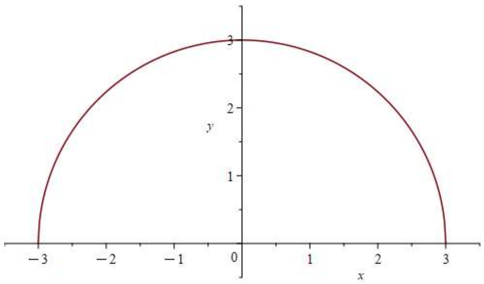


We can easily see from the drawing that we need to use polar coordinates with radius $r = 1$ and we will integrate from the red curve, that is, $x^2 + y^2 = r^2$, until the green one, that is, $\sqrt{x^2 + y^2} = r$. Therefore:

$$\int_0^{2\pi}\int_0^1 (r-r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^2 - r^3\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi}\left(\frac{1}{3}-\frac{1}{4}\right)d\theta = \frac{2\pi}{12} = \frac{\pi}{6}\, u^3$$

- Exercise 15.2.12: Compute the following integral by converting to polar coordinates.

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\ dy\, dx$$



By drawing the area of integration, we can easily determine the bounds for r and θ. We can use Maple to plot the function $y = \sqrt{9-x^2}$. From the plot on the right, we can easily see that the radius will go from $r = 0$ to $r = 3$ and the angle θ will go from 0 to π. Also, remember that $r^2 = x^2 + y^2$. Therefore:

$$\int_0^{\pi}\int_0^3 \sin(r^2)\, r\, dr\, d\theta = \int_0^{\pi}\left[-\frac{1}{2}\cos r^2\right]_{r=0}^{r=3} d\theta = -\frac{1}{2}\int_0^{\pi}\cos(9) - 1\, d\theta = \frac{\pi}{2}\,(1-\cos 9)$$