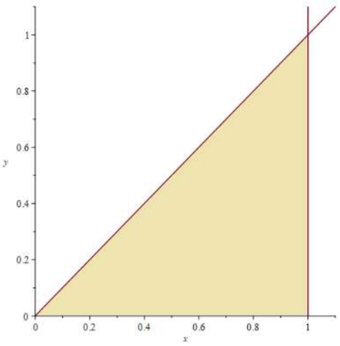
WEEK 6 – CLASS EXERCISES

EXERCISE 1: Find the surface area of the part of the surface $\boldsymbol{z = x^2 + 2y}$ that lies above the triangular region in the $\boldsymbol{xy}$-plane with vertices $(\mathbf{0},\mathbf{0})$, $(\mathbf{1},\mathbf{0})$ and $(\mathbf{1},\mathbf{1})$.

Let us consider the function $z(x,y) = x^2 + 2y$ and let's calculate its first-order partial derivatives:

$$\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad \frac{\partial z}{\partial y} = 2$$

In order to determine the bounds of the double integral, let us first plot the integration area. From the picture on the right, we can see that the x variable goes from 0 to 1 and the y values starts at $y = 0$ and is bounded from above by the line $y = x$. Now, let $\vec{n}$ be the normal vector to the surface:



$$\vec{n} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = (-2x, -2, 1)$$

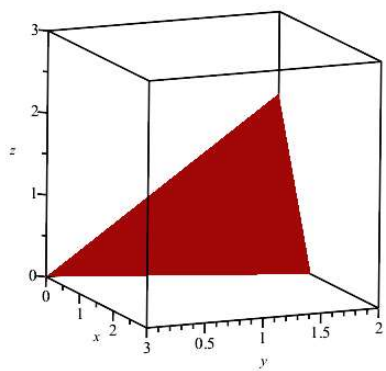
Then, the area of the surface $z(x,y)$ will be:

$$\int_0^1 \int_0^x \|\vec{n}\|\, dy\, dx = \int_0^1 \int_0^x \sqrt{4x^2 + 4 + 1}\, dy\, dx = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx = \int_0^1 \left[y\sqrt{4x^2+5} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 x\sqrt{4x^2+5}\, dx = \left[\frac{1}{12}\left(4x^2+5\right)^{3/2}\right]_0^1 = \frac{9^{3/2} - 5^{3/2}}{12} = \frac{27 - 5\sqrt{5}}{12}\, u^2$$

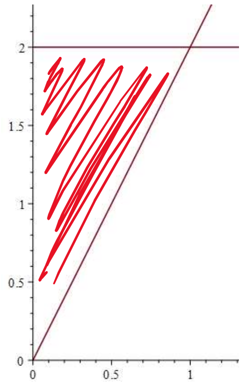
EXERCISE 2: Let $\boldsymbol{E}$ denote the solid bounded by the surfaces $\boldsymbol{z = 0}$, $\boldsymbol{x = 0}$, $\boldsymbol{y = 2}$ and $\boldsymbol{z = y - 2x}$.

$(\boldsymbol{a})$ Sketch the solid $\boldsymbol{E}$.

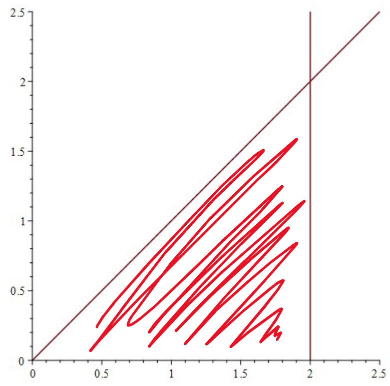


$(\boldsymbol{b})$ Sketch the projections of the solid $\boldsymbol{E}$ on the $\boldsymbol{xy}$ plane, the $\boldsymbol{yz}$ plane and the $\boldsymbol{xz}$ plane.

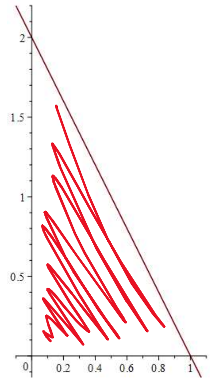
Let's use Maple in order to plot the projections of the solid on the planes. Note that the intersection of the plane $z = y - 2x$ with the xy plane $(z = 0)$ is the line $y = 2x$. Similarly, the intersections of such plane with the yz $(x = 0)$ and xz $(y = 2)$ plane are $z = y$ and $z = 2 - 2x$ respectively. Therefore:



xy plane



yz plane



xz plane

$(\boldsymbol{c})$ Express the integral

$$\iiint_E \boldsymbol{f(x,y,z)\, dV}$$

as an iterated integral in six different ways. Of course, since $\boldsymbol{f}$ is not given, you cannot evaluate these integrals.

By looking at the graphs in parts (a) and (b), we can easily determine the limits for the integrals:

$$\int_0^2 \int_z^2 \int_0^{\frac{y-z}{2}} f(x,y,z)\, dx\, dy\, dz \qquad \int_0^2 \int_0^y \int_0^{\frac{y-z}{2}} f(x,y,z)\, dx\, dz\, dy$$

$$\int_0^2 \int_0^{1-\frac{z}{2}} \int_{z+2x}^2 f(x,y,z)\, dy\, dx\, dz \qquad \int_0^1 \int_0^{2-2x} \int_{z+2x}^2 f(x,y,z)\, dy\, dz\, dx$$

$$\int_0^2 \int_0^{\frac{y}{2}} \int_0^{y-2x} f(x,y,z)\, dz\, dx\, dy \qquad \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x,y,z)\, dz\, dy\, dx$$

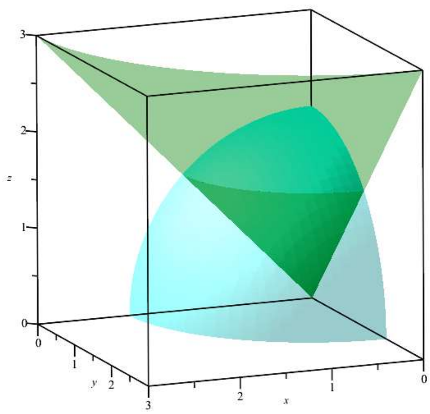
EXERCISE 3: Consider the solid region $\boldsymbol{E}$ that lies below $\boldsymbol{x^2+y^2+z^2=4}$ and above $\boldsymbol{z=\sqrt{x^2+y^2}}$ and is in the first octant, that is, $\boldsymbol{x \geq 0}$, $\boldsymbol{y \geq 0}$ and $\boldsymbol{z \geq 0}$.

$(\boldsymbol{a})$ Sketch the solid $\boldsymbol{E}$.

```
with(plots) :

a1 := implicitplot3d(x^2 + y^2 + z^2 = 4, x = 0..3, y = 0..3, z = 0..sqrt(2), colour = cyan, transparency = 0.6) :
a2 := implicitplot3d(x^2 + y^2 + z^2 = 4, x = 0..3, y = 0..3, z = sqrt(2)..3, colour = cyan, transparency = 0) :
b1 := implicitplot3d(z = sqrt(x^2 + y^2), x = 0..3, y = 0..3, z = sqrt(2)..3, colour = green, transparency = 0.6) :
b2 := implicitplot3d(z = sqrt(x^2 + y^2), x = 0..3, y = 0..3, z = 0..sqrt(2), colour = green, transparency = 0) :

display(a1, a2, b1, b2)
```



$(\boldsymbol{b})$ Find the volume of $\boldsymbol{E}$ using cylindrical coordinates.

We can determine the intersection of the two surfaces with a system of equations:

$$\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{x^2+y^2} \end{cases} \Rightarrow \quad x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=4 \quad \Rightarrow \quad 2x^2+2y^2=4 \quad \Rightarrow \quad x^2+y^2=2$$

The intersection of the two curves is therefore a circle of radius $\sqrt{2}$. Therefore, we will integrate between $r = 0$ and $r = \sqrt{2}$. Since we are in the first octant, the angle θ will go from $\theta = 0$ to $\theta = \frac{\pi}{2}$. The z coordinate is bounded from below by the equation of the cone, that is, $z = \sqrt{x^2 + y^2} = r$, and from above by $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$. Therefore, the volume of the solid E will be:

$$\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \left(r\sqrt{4 - r^2} - r^2 \right) dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[-\frac{1}{3}\,(4 - r^2)^{3/2} - \frac{r^3}{3} \right]_{r=0}^{r=\sqrt{2}} d\theta$$

$$\int_0^{\frac{\pi}{2}} \left(-\frac{2^{3/2}}{3} - \frac{2\sqrt{2}}{3} + \frac{4^{3/2}}{3} \right) d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{8}{3} - \frac{4\sqrt{2}}{3} \right) d\theta = \frac{\pi}{2} \left(\frac{8}{3} - \frac{4\sqrt{2}}{3} \right) = \frac{2\pi}{3}\,(2 - \sqrt{2})\, u^3$$

$(\boldsymbol{c})$ Find the volume of $\boldsymbol{E}$ using spherical coordinates.

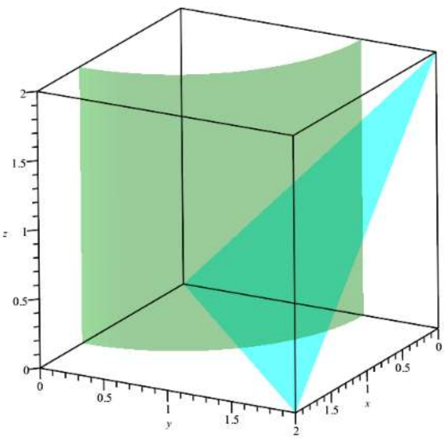
Since the angle of the cone is $\phi = \frac{\pi}{4}$ and the radius of the sphere is $\rho = 2$, then the volume of the solid E can also be calculated with the following integral:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \rho^2 \sin\phi\; d\rho\, d\phi\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sin\phi \left[\frac{\rho^3}{3} \right]_0^2 d\phi\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{8}{3} \sin\phi\; d\phi\, d\theta = \int_0^{\frac{\pi}{2}} \left[-\frac{8}{3}\cos\phi \right]_0^{\frac{\pi}{4}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{8}{3}\left(\cos\frac{\pi}{4} - \cos 0 \right) d\theta = \int_0^{\frac{\pi}{2}} -\frac{8}{3}\left(\frac{\sqrt{2}}{2} - 1 \right) d\theta = -\frac{8\pi}{6}\left(\frac{\sqrt{2}}{2} - 1 \right) = \frac{2\pi}{3}\,(2 - \sqrt{2})\, u^3$$

EXERCISE 4: Set up an integral in cylindrical coordinates to represent the volume of the region in the first octant, that is, $\boldsymbol{x \geq 0}$, $\boldsymbol{y \geq 0}$ and $\boldsymbol{z \geq 0}$, that lies above the $\boldsymbol{xy}$-plane, below the plane $\boldsymbol{z = y - x}$ and inside the cylinder $\boldsymbol{x^2 + y^2 = 4}$. Evaluate the integral.

Let's try to determine the bounds for the integral from the graph on the right. Since the cylinder has a radius of 2, we will then integrate r from 0 to 2. The range for the angle θ can be determined form the equation of the plane: since $z = y - x = r\,(\sin\theta - \cos\theta)$ and $z \geq 0$, then we have $\sin\theta - \cos\theta \geq 0$, which is the case for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. Finally, the z coordinate is bounded from above by the plane $z = y - x = r\,(\sin\theta - \cos\theta)$. Therefore, the volume of the solid will be:

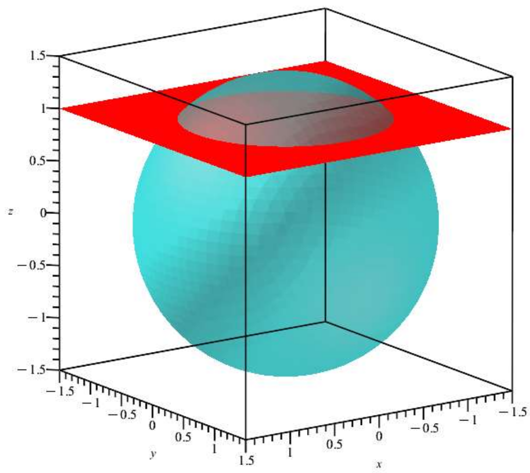


$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 \int_0^{r(\sin\theta - \cos\theta)} r\, dz\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 r^2\,(\sin\theta - \cos\theta)\, dr\, d\theta$$

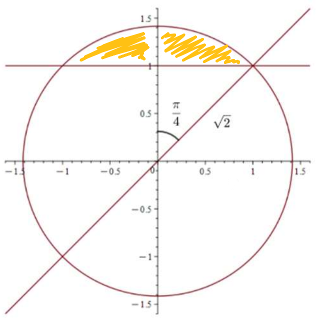
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin\theta - \cos\theta) \left[\frac{r^3}{3} \right]_{r=0}^{r=2} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{8}{3}\,(\sin\theta - \cos\theta)\, d\theta$$

$$= \left[\frac{8}{3}(-\cos\theta - \sin\theta) \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{8}{3}\left(-1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{8}{3}(\sqrt{2} - 1)\, u^3$$

EXERCISE 5: Let $\boldsymbol{E}$ be the solid region that lies inside the sphere $\boldsymbol{x^2 + y^2 + z^2 = 2}$ and above the plane $\boldsymbol{z = 1}$. Let $\boldsymbol{d(x, y, z) = z^2}$ be the density of $\boldsymbol{E}$. Sketch $\boldsymbol{E}$ and write a triple integral in spherical coordinates that gives the mass of $\boldsymbol{E}$. You do not have to evaluate the integral.



The bounds of integration for the angle θ are very straightforward: we will need to integrate from $\theta = 0$ to $\theta = 2\pi$. However, the bounds of the other integrals seem to be a bit harder. Let us then determine the intersection of the two surfaces: by substituting $z = 1$ in the equation of the sphere, we get that the curve of intersection is a circumference $x^2 + y^2 = 1$, that is, it has a radius of $r = 1$. The projection of the solid on the xz plane will then be the following:



Now, we can easily see from the drawing that we need to integrate the angle ϕ, from $\phi = 0$ until $\phi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$. Regarding the radius ρ, it is obvious that the upper bound will be determined by the radius of the sphere $\rho = \sqrt{2}$. On the other hand, the lower bound will then be determined by the equation of the plane:

$$z = \rho\cos\phi = 1 \quad \Longrightarrow \quad \rho = \frac{1}{\cos\phi}$$

Also, we can rewrite the density function in spherical coordinates as: $d(\rho, \theta, \phi) = z^2 = (\rho\cos\phi)^2$

Therefore, the volume of the solid region E can be calculated as:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\phi}}^{\sqrt{2}} d(\rho, \theta, \phi)\, \rho^2 \sin\phi \;\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{co}}^{\sqrt{2}} (\rho\cos\phi)^2\, \rho^2 \sin\phi \;\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\phi}}^{\sqrt{2}} \rho^4 \cos^2\phi \sin\phi \;\, d\rho\, d\phi\, d\theta$$