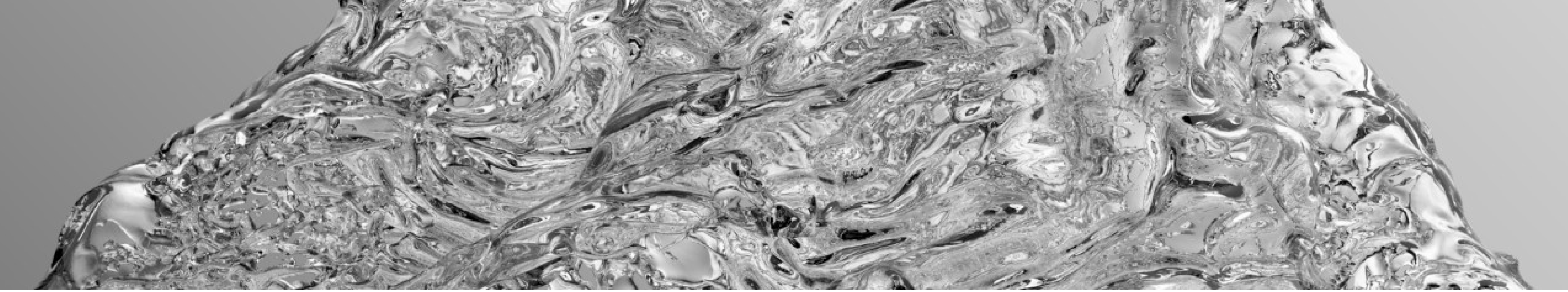


EEN-E2001 Computational Fluid Dynamics

Lecture 4: Navier-Stokes equation and pressure

Prof. Ville Vuorinen

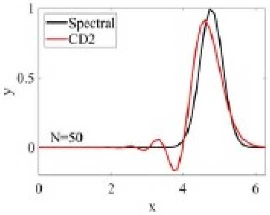
February 6th 2023

Aalto University, School of Engineering

Lecture 1: Linear PDEs and finite difference method

$$\frac{\partial T}{\partial t}+\nabla\cdot T\boldsymbol{u}=\alpha\nabla^2 T$$

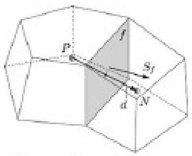
$$\frac{\partial T}{\partial x}\approx\frac{T_{i+1}-T_{i-1}}{2\Delta x}$$



Lecture 2: Gauss' theorem and finite volume method

$$\int_\Omega \nabla\cdot(T\boldsymbol{u})\,d\Omega=\int_{\partial\Omega}(T\boldsymbol{u})\cdot\boldsymbol{n}\,dS$$

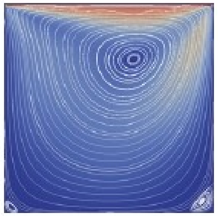
$$\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{n}\,dS\approx\Sigma_f\boldsymbol{u}_f\cdot\boldsymbol{n}_f\,dS_f$$



Lecture 3: Navier-Stokes equation and pressure

$$\frac{\partial\boldsymbol{u}}{\partial t}+\nabla\cdot\boldsymbol{u}\boldsymbol{u}=-\nabla p+\nu\nabla^2\boldsymbol{u}$$

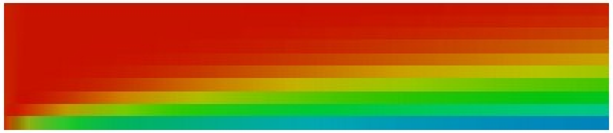
$$-\nabla^2 p=\nabla\cdot\nabla\cdot\boldsymbol{u}\boldsymbol{u}$$

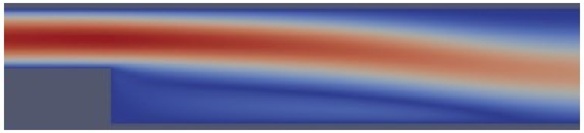


Lecture 4: OpenFOAM code and structure

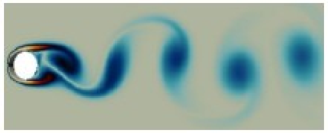
```
fvVectorMatrix UEqn
(
      fvm::ddt(U)
    + fvm::div(phi, U)
    - fvm::laplacian(nu, U)
);
```

Lecture 5: Simulating fluid physical phenomena: part A





Lecture 6: Simulating fluid physical phenomena: part B



Intended learning objectives of the full lecture

After the lecture the student:

- Can explain what is the role of pressure in incompressible flows.

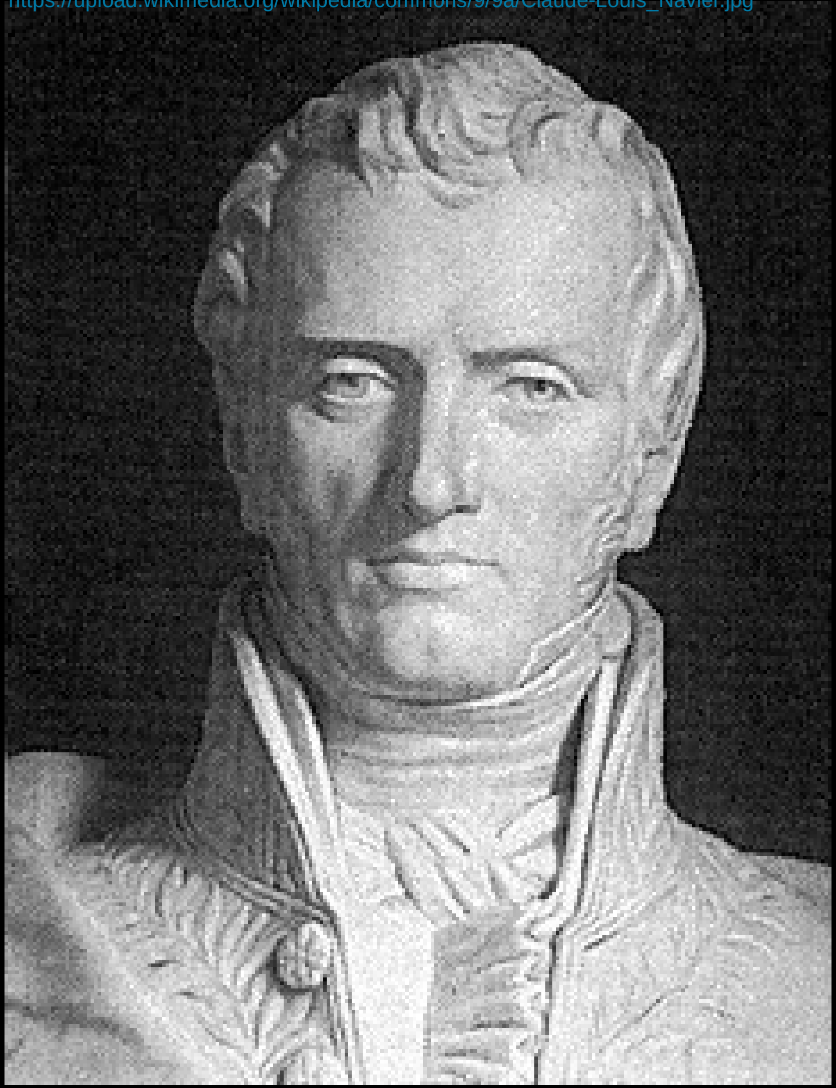
- Can explain the basic idea how Navier-Stokes equation is discretized in fvm using projection method and explicit time integration.

Navier-Stokes equations

Clay Institute: Among the 7 most important unsolved problems in math

https://en.wikipedia.org/wiki/Navier–Stokes_equations

https://upload.wikimedia.org/wikipedia/commons/9/9a/Claude-Louis_Navier.jpg



https://upload.wikimedia.org/wikipedia/commons/thumb/a/ad/Ggstokes.jpg/300px-Ggstokes.jpg

CFD simulation and PDE solution includes at least the following aspects covered on the course

1) **Physics** identification.

2) **Mathematical equations and physics interpretation.** Boundary/initial conditions.

3) **Objectives, feasibility, and time-constraints**.

4) **Numerical method and modeling assumptions**.

5) **Geometry and mesh generation**.

6) **Computing** i.e. running simulation.

7) **Visualization and post-processing**.

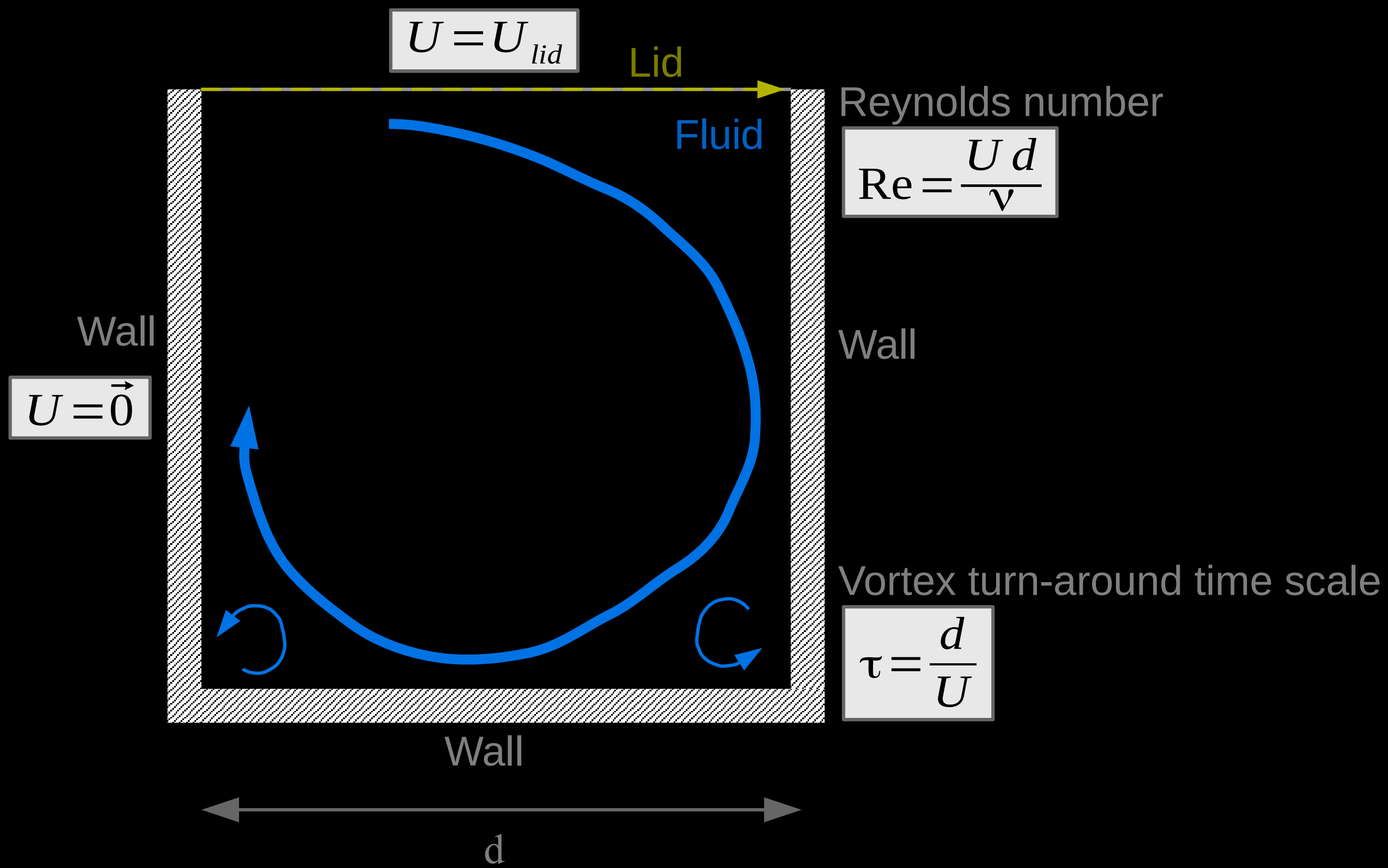
8) **Validation and verification, reference data**. Reporting, analysis and discussion of the results. Are the results sane?

Motivational aspects to study flow in a cavity

Relevance: HW2

Homework 2: Lid driven cavity

Moving lid creates a vortex in a box



Motivation: window induced airflow in a car is a simple example of "flow in a cavity"

Mathai et al. (2021)
https://www.science.org/doi/epdf/10.1126/sciadv.abe0166

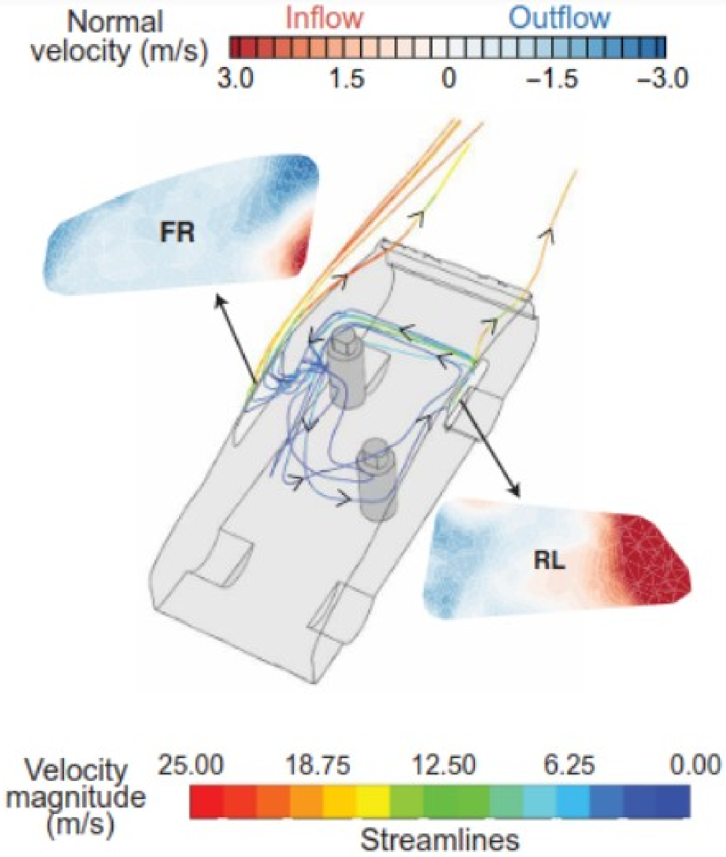
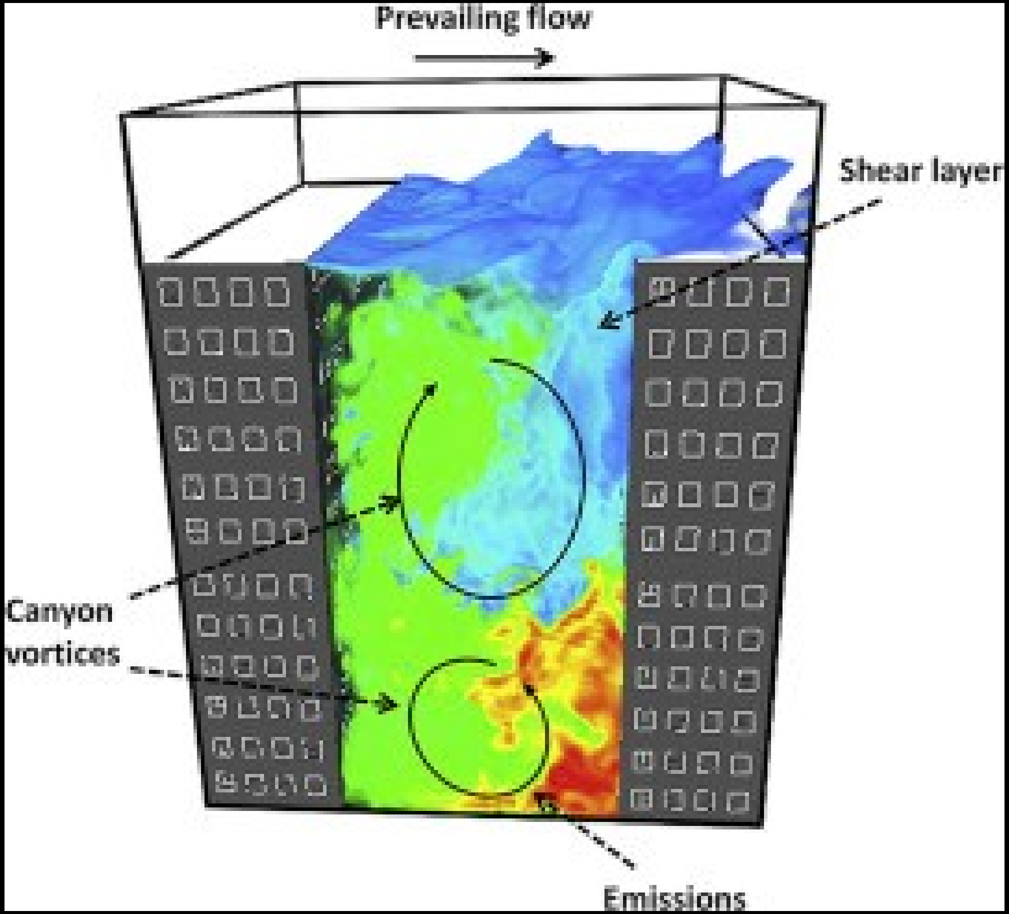


Fig. 3. Streamlines computed for the case in which the RL and FR windows are open. The streamlines were initiated at the RL window opening. The streamline color indicates the flow velocity. Insets show the FR and RL windows colored by the normal velocity. The RL window has a strong inflow (positive) of ambient air, concentrated at its rear, whereas the FR window predominantly shows an outward flow (negative) to the ambient.

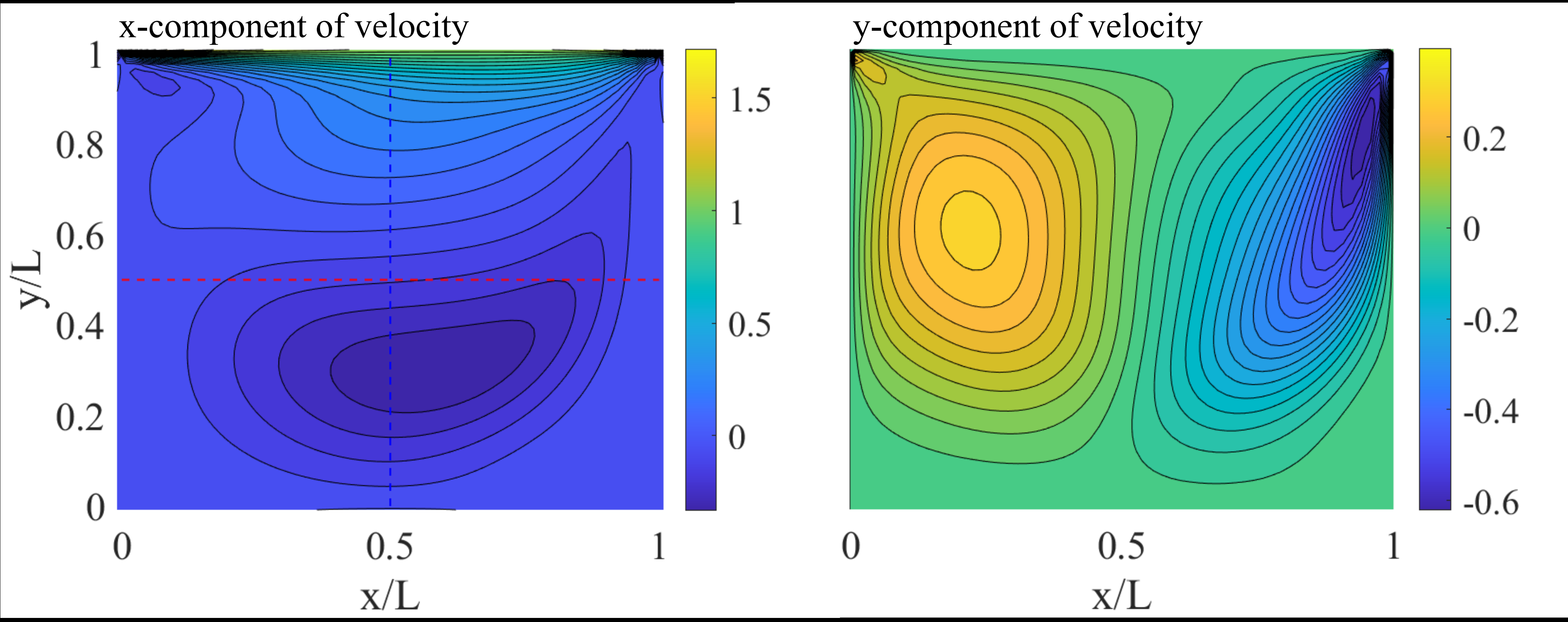
Motivation: another example for "flow in a cavity" could be wind flow pattern between two buildings

Zhong et al. (2017)
https://www.sciencedirect.com/science/article/abs/pii/S0269749116312398#undfig1



Homework 2: Lid driven cavity

Comparison of a Navier-Stokes simulation against reference data by Ghia et al. Here: Re=100 & resolution **60^2** points.



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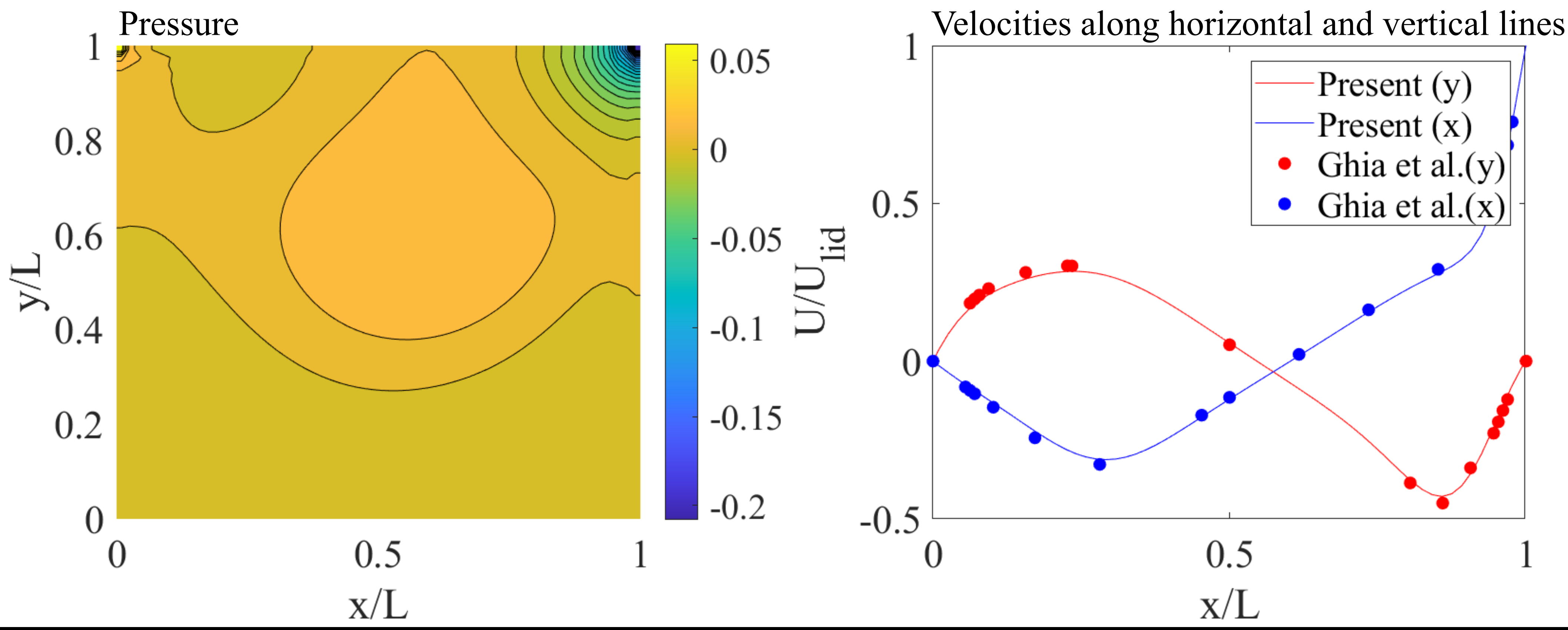
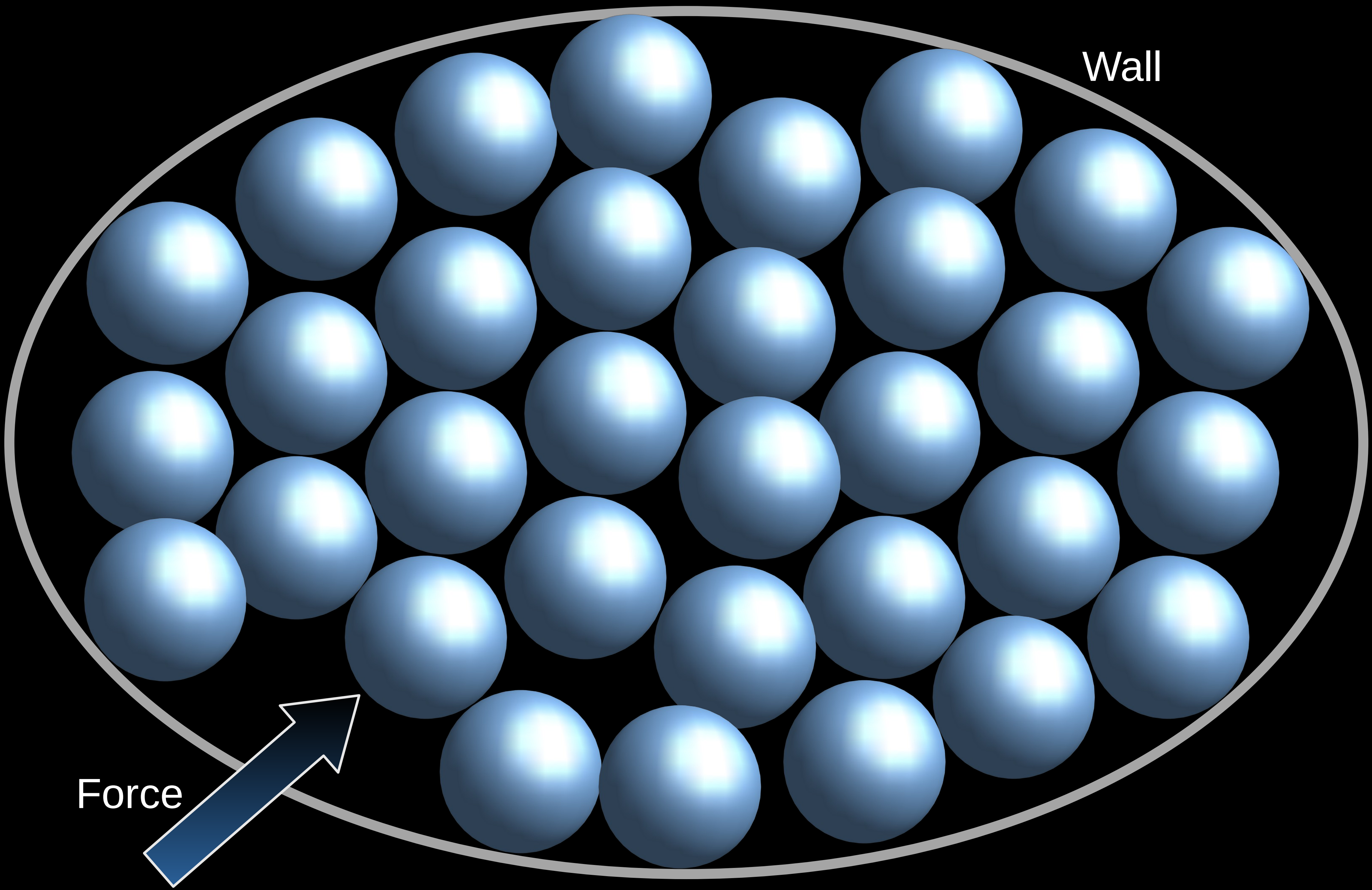
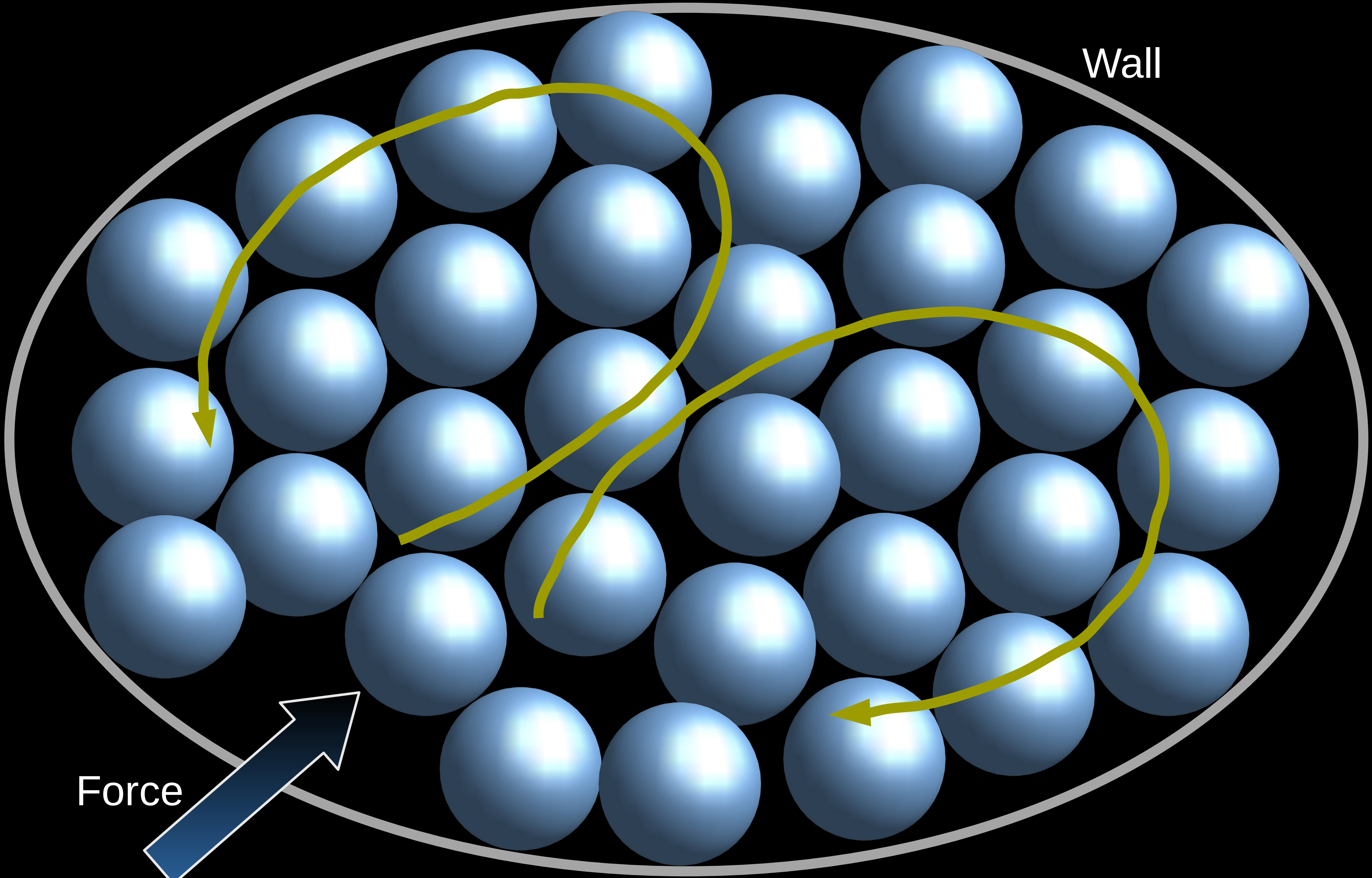


Incompressible Navier-Stokes equation

Rigid sphere analogy and incompressible flows: what happens when one of the spheres is pushed in a box?



A vortical motion appears when all other spheres "feel" the force while the wall keeps the spheres confined in the container. This is called "elliptical" or "global" character. In incompressible flows pressure is solved from Poisson eqn which is of such elliptical character: when fluid element moves all other elements will immediately feel that motion via the pressure changes.



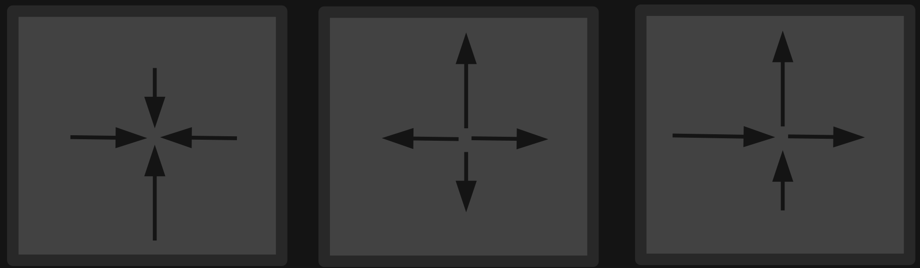
Compressible and incompressible flows

Continuity equation tells that mass is conserved (ρ = density):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{u}\rho) = 0$$

Velocity vectors in a given point

$\frac{\partial \rho}{\partial t} > 0, \nabla \cdot (\vec{u}\rho) < 0$ | $\frac{\partial \rho}{\partial t} < 0, \nabla \cdot (\vec{u}\rho) > 0$ | $\frac{\partial \rho}{\partial t} = 0, \nabla \cdot \vec{u} = 0$



Compression | Expansion | Incompressible

The incompressibility condition: at each point massflow entering a control volume = mass flow exiting the control volume (ρ = constant).

$$\nabla \cdot \vec{u} = 0$$

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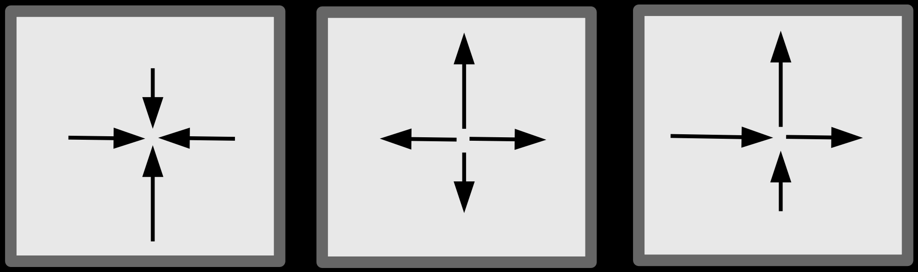

Compression Expansion Incompressible

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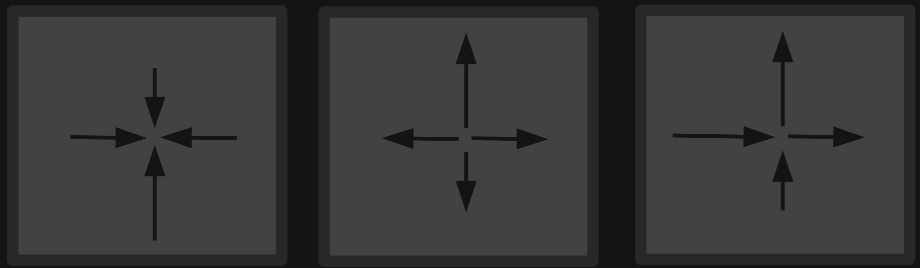


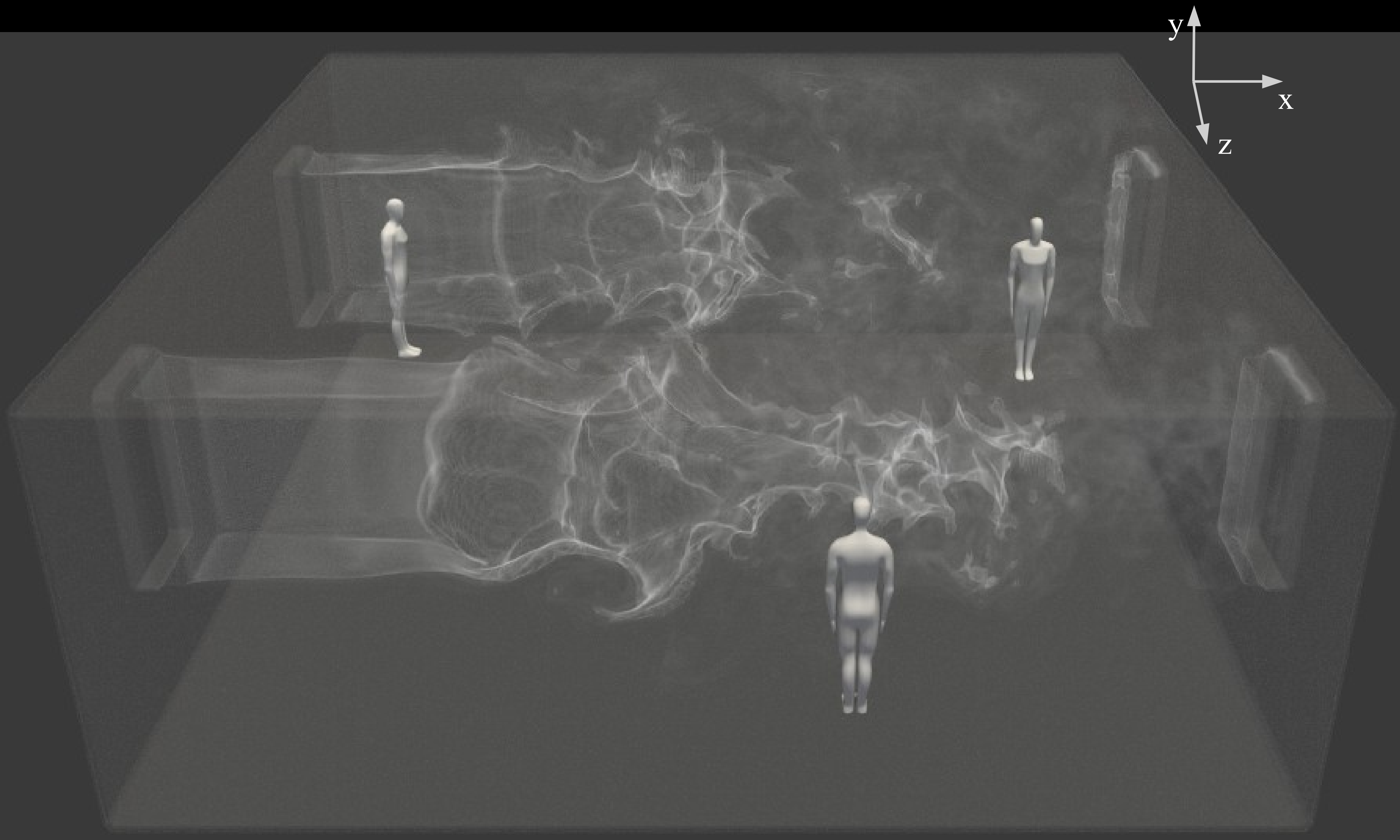
Compression Expansion Incompressible

The incompressibility condition: at each point massflow entering a control volume = mass flow exiting the control volume (ρ = constant).

$$\nabla \cdot \vec{u} = 0$$

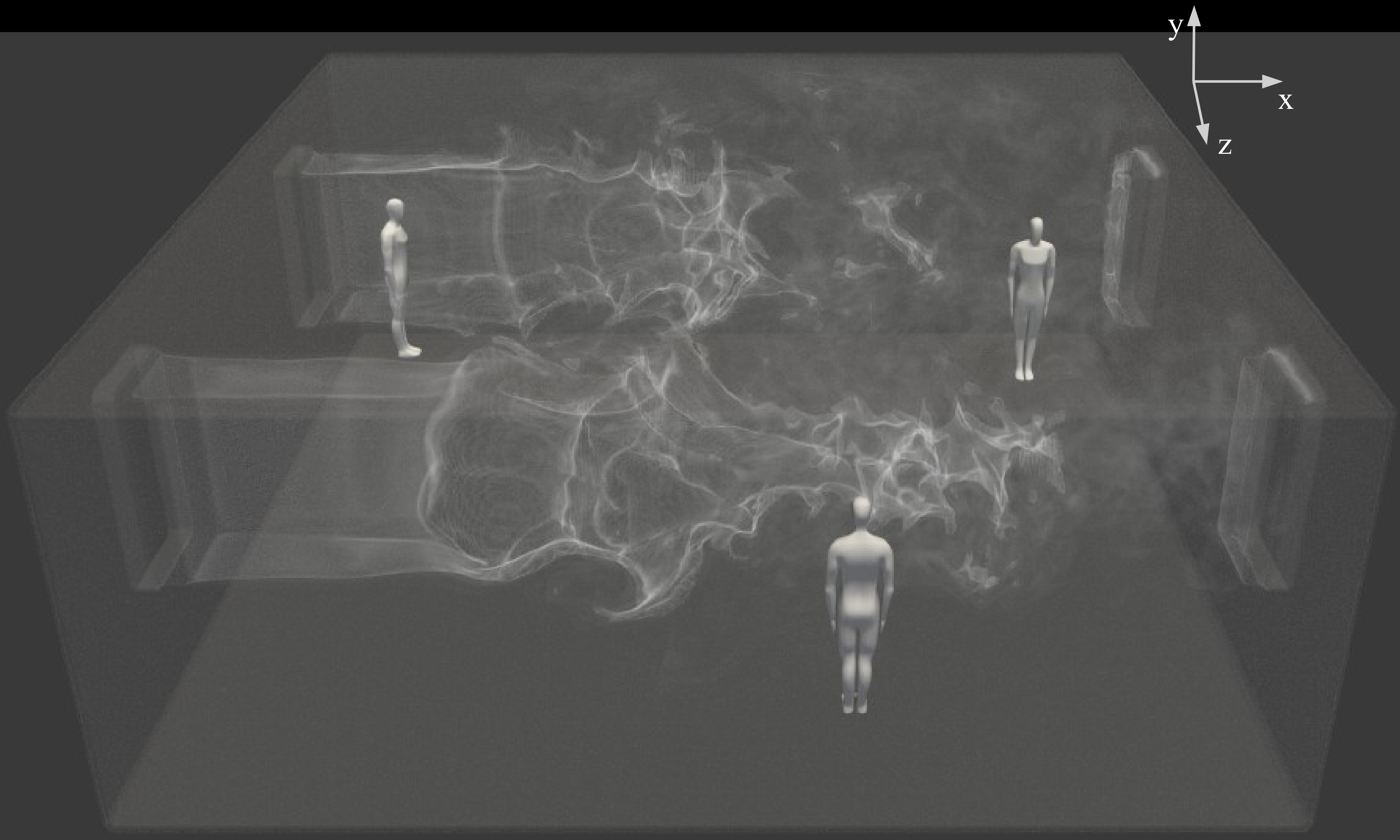
Gauss theorem applied on the mass conservation law in a cross-draught situation (room with four windows):

$$\int_{V_{room}} \nabla \cdot \vec{u}\, dV = \int_{A_{room}} \vec{u} \cdot \boldsymbol{n}\, dA = 0$$



The standard mass conservation law is commonly written on B.Sc./M.Sc. courses as $U_1A_1 + U_2A_2 = U_3A_3 + U_4A_4$. Since $-\mathbf{n}_{\text{left}} = \mathbf{n}_{\text{right}}$ we get:

$$\int_{A_{win1}} \vec{u}\cdot\boldsymbol{n}\,dA + \int_{A_{win2}} \vec{u}\cdot\boldsymbol{n}\,dA = \int_{A_{win3}} \vec{u}\cdot\boldsymbol{n}\,dA + \int_{A_{win4}} \vec{u}\cdot\boldsymbol{n}\,dA$$



Incompressible Navier-Stokes equation

Convection form

i^{th} component of velocity

Kinematic viscosity [m²/s]

$$\frac{\partial u_i}{\partial t} + \vec{u} \cdot \nabla u_i = -\nabla p + \nu \nabla^2 u_i$$

Pressure

Note: essentially same eqn

Convection-diffusion eqn

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \nabla c = \alpha \nabla^2 c$$

$$\nabla \cdot \vec{u} = 0$$

Incompressible Navier-Stokes equation

Conservative form

i^{th} component of velocity

Kinematic viscosity [m²/s]

$$\frac{\partial u_i}{\partial t} + \nabla \cdot \vec{u}\, u_i = -\nabla p + \nu \nabla^2 u_i$$

Pressure

Convection-diffusion eqn

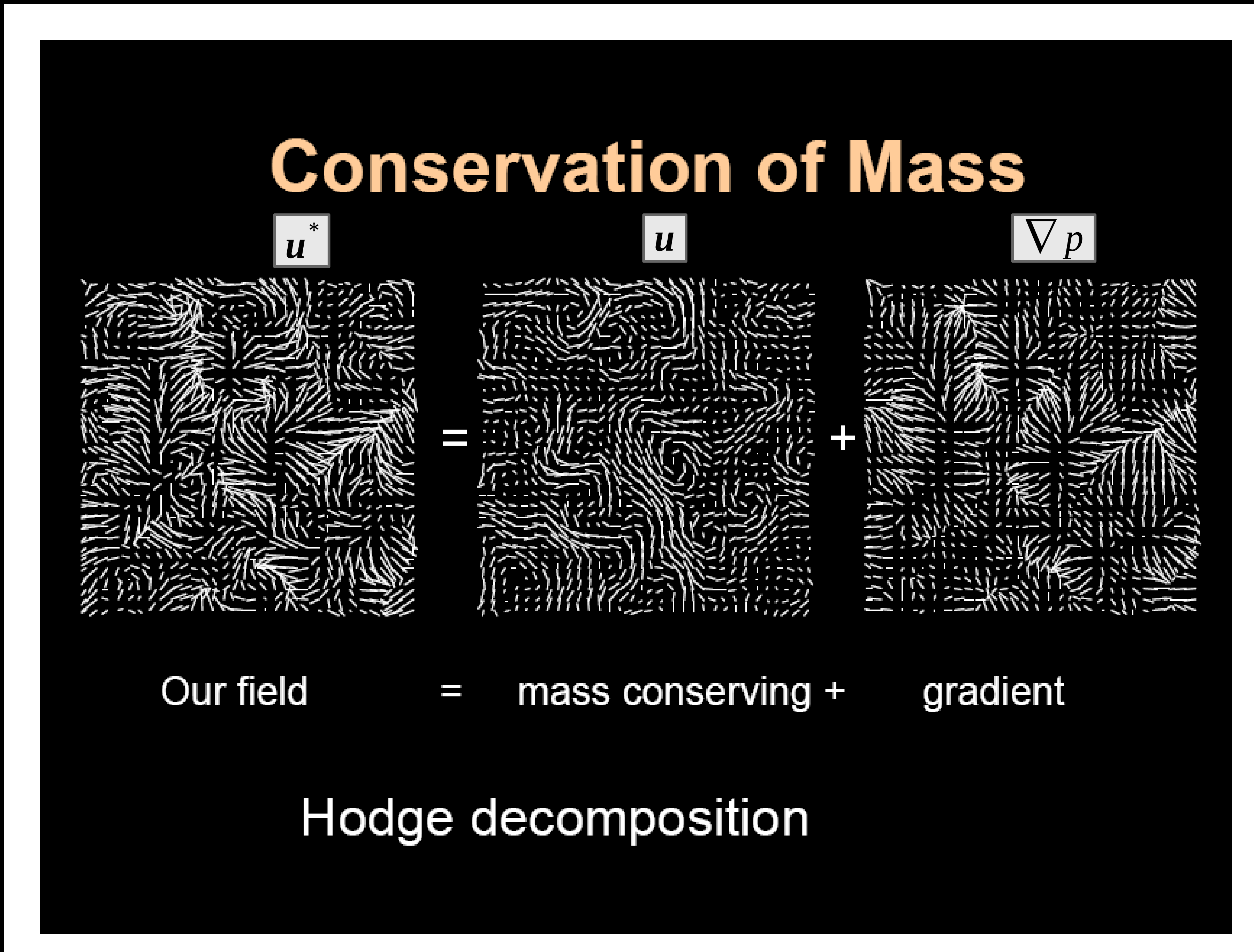
$$\frac{\partial c}{\partial t} + \nabla \cdot \vec{u}\, c = \alpha \nabla^2 c$$

$$\nabla \cdot \vec{u} = 0$$

Pressure in incompressible flows

Relevance: Non-linear convection term creates divergence to velocity which must be “removed” by pressure gradient to obtain mass balance and fulfill $\mathrm{div}(\mathbf{u}) = 0$ condition

Vector fields can be divided into two parts via “Helmholtz-Hodge” decomposition



Mathematical background for the projection method

Vectors: C=convection term, D=diffusion term, P=pressure gradient

The core problem in solving incompressible Navier-Stokes method:
We would like to find a $\Delta u_i = \Delta t(-C+D+P)$ to update solution at each grid point as $u_i^{n+1} = u_i^n + \Delta u_i$. We can estimate $C=C(x,y,z,t)$ & $D=D(x,y,z,t)$ $(t=n\Delta t)$. But how do we know the pressure gradient $P=P(x,y,z,t)$?

Helmholtz-Hodge theorem: any vector field $\mathbf{u}^*$ can be expressed as a sum of fields as follows (B.Sc. math):

$$\boldsymbol{u}^* = \boldsymbol{u} + \nabla p \text{, where } \nabla \cdot \boldsymbol{u} = 0 \text{ and } \nabla \times \nabla p = 0$$

Consequence of Helmholtz-Hodge theorem: the function p is a solution of the Poisson eqn. If we could find $\mathbf{u}^*$ then we can also solve p.

$$\nabla^2 p = \nabla \cdot \boldsymbol{u}^*$$

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Derivation of Poisson equation for pressure 1/2

$$\frac{\partial u_i}{\partial t}+\frac{\partial u_j u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j^2}$$

Take divergence from both sides of NS-eqn

$$\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_i}+\frac{\partial}{\partial x_i}\frac{\partial u_j u_i}{\partial x_j}=-\frac{\partial^2 p}{\partial x_i^2}+\nu\frac{\partial^2}{\partial x_j^2}\frac{\partial u_i}{\partial x_i}$$

Derivation of Poisson equation for pressure 2/2

Incompressible flow
→ div(**u**)=0

Incompressible flow
→ div(**u**)=0

$$\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_i}\frac{\partial u_j u_i}{\partial x_j} = -\frac{\partial^2 p}{\partial x_i^2} + \nu \frac{\partial^2}{\partial x_j^2}\frac{\partial u_i}{\partial x_i}$$

Poisson equation for pressure

$$-\frac{\partial^2 p}{\partial x_i^2} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$$-\nabla^2 p = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

In practice Poisson equation is a matrix equation which is solved with a linear system solver for A x = b (more later in the course).

Projection method in a nutshell

Projection method

Step 1: Calculate velocity prediction **u*** using only **C** and **D** from previous timestep

$$\boldsymbol{u}^* = \boldsymbol{u}_n + \Delta t (D - C)$$

Step 2: Calculate divergence of u* and use it as source term in Poisson equation to find p using a standard Poisson solver.

$$\nabla^2 p = \nabla \cdot \boldsymbol{u}^*$$

Step 3: Calculate pressure gradient via e.g. CD2 and do pressure correction (projection)

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}^* - \nabla p$$

Step 4: Based on Helmholz-Hodge theorem $\mathbf{u}_{n+1}$ is incompressible and mass conserving:

$$\nabla \cdot \boldsymbol{u}_{n+1} = 0$$

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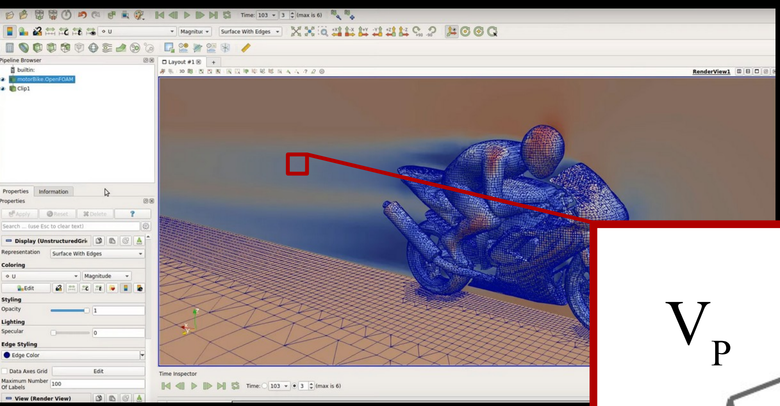
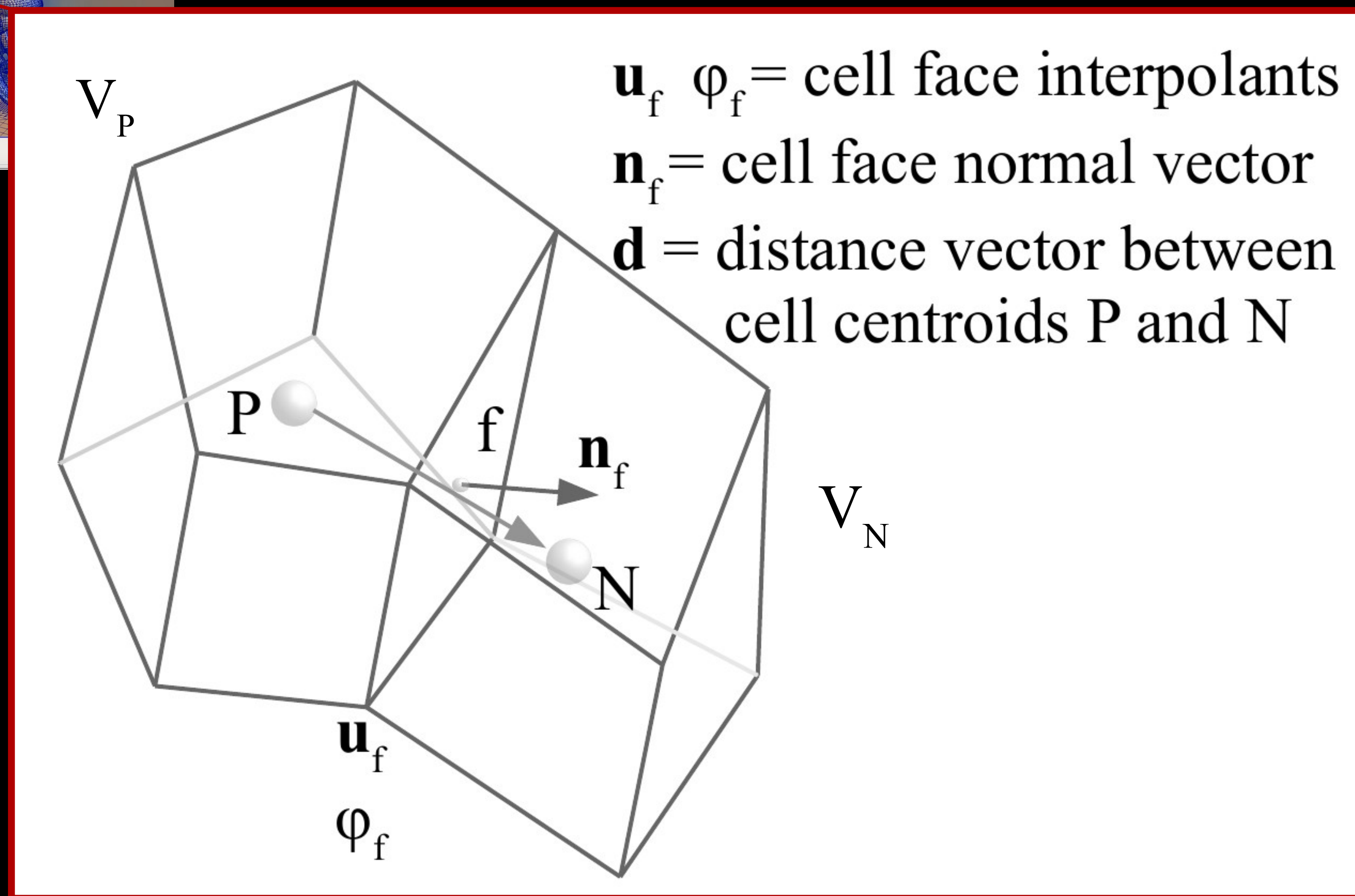
Basic idea of finite volume method: divide geometry into small volumes and update numerical solution at volume centroids by estimating e.g. mass & momentum fluxes through the faces of the control volumes during small time intervals



Control volume

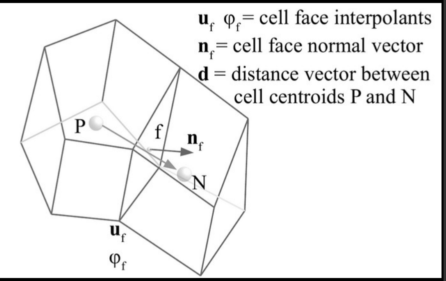


Finite volume method in a nutshell for Navier-Stokes eqn and projection method. Discretization for Steps 1-3 is discussed below:

Step 1: fvm estimation of convection $C_i=C_i(x,y,z,t)$ and diffusion $D_i=D_i(x,y,z,t)$ terms in Navier-Stokes eqn to find the prediction u^* $(t=n\Delta t)$:

$$C_i=\frac{1}{V}\int_V \nabla\cdot(\boldsymbol{u}\,u_i)\,dV=\frac{1}{V}\int_A (\boldsymbol{u}\,u_i)\cdot\boldsymbol{n}\,dA\approx\frac{1}{V}\Sigma_{faces}(\boldsymbol{u_f}\,u_f^i)\cdot\boldsymbol{n_f}\,dA_f$$

$$D_i=\frac{1}{V}\int_V \nu\nabla\cdot\nabla u_i\,dV=\frac{1}{V}\int_A \nu\nabla u_i\cdot\boldsymbol{n}\,dA\approx\frac{1}{V}\Sigma_{faces}\,\nu\nabla u_f^i\cdot\boldsymbol{n_f}\,dA_f$$



Step 2a: fvm estimation of divergence of u*

$$\nabla\cdot\boldsymbol{u}^*\approx\frac{1}{V}\int_V \nabla\cdot\boldsymbol{u}^*\,dV\approx\frac{1}{V}\Sigma_{faces}\,\boldsymbol{u_f}\cdot\boldsymbol{n_f}\,dA_f$$

Step 2b: fvm estimation of matrix eqn for pressure (more on matrices later on)

$$\Delta p\approx\frac{1}{V}\int_V \nabla\cdot\nabla p\,dV=\frac{1}{V}\int_A \nabla p\cdot\boldsymbol{n}\,dA\approx\frac{1}{V}\Sigma_{faces}\,\nabla p_f\cdot\boldsymbol{n_f}\,dA_f$$

Step 3: fvm estimation of pressure gradient

$$\nabla p\approx\frac{1}{V}\int_V \nabla p\,dV=\frac{1}{V}\int_A p\,n_i\,dA_i\approx\frac{1}{V}\Sigma_{faces}\,p_f^i\,n_f^i\,dA_f^i$$