MS-C1081 – Abstract Algebra **Problem set 2**
2022–2023 (Period III)
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During the exercise sessions you may work on the Warm-up problems, and the TAs will present solutions to those. Submit your solutions to Homework 1,2,3 and to the Fill-in-the-blanks problem on MyCourses before the deadline. Remember that the Homeworks and the Fill-in-the-blanks go to separate return boxes, and each return box accepts a single pdf file.

Warm-ups

Warm-up 1. Consider the group $\mathbb{Z}$.

- If possible, for each condition in the definition of a subgroup, give examples of subsets of $\mathbb{Z}$ that fail that condition but satisfy the other two.
- If possible, for each condition in the definition of a subgroup, give examples of subsets of $\mathbb{Z}$ that satisfy that condition but fail the other two.
- If possible, give examples of homomorphisms $\mathbb{Z} \to \mathbb{Z}$ that are: injective but not surjective / surjective but not injective / neither.

Warm-up 2. Determine which of the following are group homomorphisms:

$$\begin{array}{cccc}
\mathbb{Z} \longrightarrow \mathbb{Z} & \mathbb{C} \longrightarrow \mathbb{R} & \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{R}\setminus\{0\} & \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R}) \\
n \longmapsto 2n & z \longmapsto |z| & z \longmapsto |z| & A \longmapsto A^{-1} \\[1em]
\mathbb{Z} \longrightarrow \mathbb{Z} & \mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\} & \mathbb{Q}\setminus\{0\} \longrightarrow \mathbb{Q}\setminus\{0\} & \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R}) \\
n \longmapsto n^2 & x \longmapsto 10^x & x \longmapsto 2x & A \longmapsto A^T \\[1em]
\mathbb{C}\setminus\{0\} \longrightarrow \mathbb{R} & \mathbb{C} \longrightarrow \mathbb{C} & \mathbb{Q}\setminus\{0\} \longrightarrow \mathbb{Q}\setminus\{0\} & \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R}) \\
z \longmapsto |z| & z \longmapsto \overline{z} & x \longmapsto x^3 & A \longmapsto (A^{-1})^T
\end{array}$$

where $\overline{z}$ is the conjugate of $z \in \mathbb{C}$, and A^T is the transpose of $A \in \mathrm{GL}_n(\mathbb{R})$. (The operation is either the usual sum or product, depending on which one turns the given set into a group.)

Warm-up 3. Let $(G, \cdot)$ be a cyclic group. Prove that G is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$, for a suitable n.
Hint: You may use Proposition 1.65 from the lecture notes.

Homework

Homework 1. Let G be a group and $H \subseteq G$ a subgroup. Prove that the following conditions are equivalent:

1. For all $a \in G$ and $b \in H$, we have $a^{-1}ba \in H$.
2. For all $x, y \in G$, if $xy \in H$, then $yx \in H$.

(Subgroups that satisfy these conditions are called *normal subgroups*.) [6 points]

Homework 2. Let G be a group. For all $a \in G$, consider the map

$$C_a\colon G \longrightarrow G$$
$$x \longmapsto a^{-1}xa.$$

Recall that the set $\mathrm{Aut}(G)$ of all automorphisms of G (i.e., isomorphisms $G \to G$) is a group with respect to composition.

1. Prove that, for all $a \in G$, the map C_a is an automorphism of G. [3 points]
2. Prove that the map

$$C\colon G \longrightarrow \mathrm{Aut}(G)$$
$$a \longmapsto C_a$$

is a group homomorphism. [3 points]

Homework 3. Given a group $(G, *_G)$, we define a group structure on the Cartesian product $G \times G = \{(x,y) \mid x, y \in G\}$, by setting

$$(x,y) * (x',y') := (x *_G x', y *_G y')$$

for all $(x,y),(x',y') \in G \times G$. The group $(G \times G, *)$ is usually simply denoted G^2 or $G \times G$. Denote by 1 the identity of G. Define

$$H := \{(x,y) \in G \times G \mid y = 1\} \qquad \text{and} \qquad K := \{(x,y) \in G \times G \mid x = y\}.$$

1. Show that $(1,1)$ is the identity of $G \times G$, and $(x,y)^{-1} = (x^{-1}, y^{-1})$. [2 points]
2. Show that H is a subgroup of $G \times G$, and H is isomorphic to G. [2 points]
3. Show that K is a subgroup of $G \times G$, and K is isomorphic to G. [2 points]

Fill-in-the-blanks. Complete the proof of the following claim:

Claim. Let G be a cyclic group and let $f\colon G \to G$ be a group homomorphism. If f is injective but not surjective, then $G \cong \mathbb{Z}$.

Proof. By ________________ of this problem set, we know that there are two cases:

$$G \cong \begin{cases} \rule{3cm}{0.4pt} \\ \rule{3cm}{0.4pt}. \end{cases}$$

We wish to show that the second case cannot happen. That is, we show that, for any $n \in \mathbb{Z}_{>0}$, there is no homomorphism from ________________ to itself that is injective and not surjective. This is a simple set-theoretic matter: more generally, there is no function that is simultaneously injective and not surjective between ________________ sets of the same cardinality. □

[3 points]