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MICROSCOPIC PHYSICS OF QUANTUM LIQUIDS

There are two ways to study quantum liquids:

(1) The fully microscopic treatment. This can be realized completely either by numerical simulations of the many-body problem, or for some special ranges of the material parameters analytically, for example, in the limit of weak interaction between the particles.

(2) A phenomenological approach in terms of effective theories. The hierarchy of the effective theories corresponds to the low-frequency, long-wavelength dynamics of quantum liquids in different ranges of frequency. Examples of effective theories: Landau theory of the Fermi liquid; Landau–Khalatnikov two-fluid hydrodynamics of superfluid ^{4}He; the theory of elasticity in solids; the Landau–Lifshitz theory of ferro- and antiferromagnetism; the London theory of superconductivity; the Leggett theory of spin dynamics in superfluid phases of ^{3}He; effective quantum electrodynamics arising in superfluid ^{3}He-A; etc. The last example indicates that the existing Standard Model of electroweak and strong interactions, and the Einstein gravity too, are the phenomenological effective theories of high-energy physics which describe its low-energy edge, while the microscopic theory of the quantum vacuum is absent.

3.1 Theory of Everything in quantum liquids

3.1.1 *Microscopic Hamiltonian*

The microscopic Theory of Everything for quantum liquids and solids – 'a set of equations capable of describing all phenomena that have been observed' (Laughlin and Pines 2000) in these quantum systems – is extremely simple. On the 'fundamental' level appropriate for quantum liquids and solids, i.e. for all practical purposes, the ^{4}He or ^{3}He atoms of these quantum systems can be considered as structureless: the ^{4}He atoms are the structureless bosons and the ^{3}He atoms are the structureless fermions with spin 1/2. The simplest Theory of Everything for a collection of a macroscopic number N of interacting ^{4}He or ^{3}He atoms is contained in the non-relativistic many-body Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial \mathbf{r}_i^2} + \sum_{i=1}^{N}\sum_{j=i+1}^{N} U(\mathbf{r}_i - \mathbf{r}_j) \ , \tag{3.1}$$

acting on the many-body wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots)$. Here m is the bare mass of the atom; $U(\mathbf{r}_i - \mathbf{r}_j)$ is the pair interaction of the bare atoms i and j.

The many-body physics can be described in the second quantized form, where the Schrödinger many-body Hamiltonian (3.1) becomes the Hamiltonian of the quantum field theory (Abrikosov *et al.* 1965):

$$\mathcal{H}-\mu\mathcal{N} = \int d\mathbf{x}\psi^\dagger(\mathbf{x}) \left[-\frac{\nabla^2}{2m} - \mu\right] \psi(\mathbf{x}) + \frac{1}{2}\int d\mathbf{x}d\mathbf{y}U(\mathbf{x}-\mathbf{y})\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x}). \tag{3.2}$$

In ^{4}He, the bosonic quantum fields $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ are the creation and annihilation operators of the ^{4}He atoms. In ^{3}He, $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ are the corresponding fermionic quantum fields and the spin indices must be added. Here $\mathcal{N} = \int d\mathbf{x}\ \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ is the operator of the particle number (number of atoms); μ is the chemical potential – the Lagrange multiplier introduced to take into account the conservation of the number of atoms. Note that the introduction of creation and annihilation operators for helium atoms is the formal procedure: it does not imply that we really can create atoms from the vacuum: this is certainly highly prohibited since the relevant energies in the liquid are of order 10 K, which is many orders of magnitude smaller than the GeV energy required to create the atom-antiatom pair from the vacuum.

3.1.2 *Particles and quasiparticles*

In quantum liquids, the analog of the quantum vacuum – the ground state of the quantum liquid – has a well-defined number of atoms. Existence of bare particles (atoms) comprising the quantum vacuum of quantum liquids represents the main difference from the relativistic quantum field theory (RQFT). In RQFT, particles and antiparticles which can be created from the quantum vacuum are similar to quasiparticles in quantum liquids. What is the analog of the bare particles – 'atoms' of the quantum vacuum of RQFT – is not clear today. At the moment we simply do not know the structure of the vacuum, and whether it is possible to describe it in terms of some discrete elements – bare particles – whose number is conserved.

In the limit when the number N of bare particles in the vacuum is large, one might expect that the difference between two quantum field theories, with and without conservation of particle number, disappears completely. However, this is not so. We shall see that the mere fact that there is a conservation law for the number of 'atoms' of the vacuum leads to a definite conclusion on the value of the relevant vacuum energy : it is exactly zero in equilibrium (see also the recent paper by Klinkhamer and Volovik (2008)). Also, as we shall see below in Chapter 29, the discreteness of the quantum vacuum can be revealed in the mesoscopic Casimir effect.

3.1.3 *Microscopic and effective symmetries*

The Theory of Everything (3.2) has a very restricted number of symmetries: (i) The Hamiltonian is invariant under translations and $SO(3)$ rotations in 3D space. (ii) There is a global $U(1)_N$ group originating from the conservation of the number N of atoms: $\mathcal{H}$ is invariant under global gauge rotation $\psi(\mathbf{x}) \to e^{i\alpha}\psi(\mathbf{x})$

with constant α. The particle number operator serves as the generator of the gauge rotations: $e^{-i\alpha\mathcal{N}}\psi e^{i\alpha\mathcal{N}} = \psi e^{i\alpha}$. (iii) In ^{3}He, the spin–orbit coupling is relatively weak. If it is ignored, then $\mathcal{H}$ is also invariant under separate rotations of spins, $SO(3)_{\mathbf{S}}$ (later we shall see that the symmetry violating spin–orbit interaction plays an important role in the physics of fermionic and bosonic zero modes in all of the superfluid phases of ^{3}He). At low temperature the phase transition to the superfluid or to the quantum crystal state occurs where some of these symmetries are broken spontaneously.

In the ^{3}He-A state all of the symmetries of the Hamiltonian, except for the translational one, are broken. However, when the temperature and energy decrease further the symmetry becomes gradually enhanced in agreement with the anti-Grand-Unification scenario (Froggatt and Nielsen 1991; Chadha and Nielsen 1983). At low energy the quantum liquid or solid is well described in terms of a dilute system of quasiparticles. These are bosons (phonons) in ^{4}He and fermions and bosons in ^{3}He, which move in the background of the effective gauge and/or gravity fields simulated by the dynamics of the collective modes. In particular, as we shall see below, phonons propagating in the inhomogeneous liquid are described by the effective Lagrangian for the scalar field α in the presence of the effective gravitational field:

$$L_{\rm effective} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\alpha\partial_\nu\alpha \ . \tag{3.3}$$

Here $g^{\mu\nu}$ is the effective acoustic metric provided by the inhomogeneity of the liquid and by its flow (Unruh 1981, 1995; Stone 2000*b* and the review by Barcelo *et al* 2005).

These quasiparticles serve as the elementary particles of the low-energy effective quantum field theory. They represent the analog of matter. The type of the effective quantum field theory – the theory of interacting fermionic and bosonic quantum fields – depends on the universality class emerging in the low-energy limit. In normal Fermi liquids, the effective quantum field theory describing dynamics of fermion zero modes in the vicinity of the Fermi suface which interact with the collective bosonic fields is the Landau theory of Fermi liquid. In superfluid ^{3}He-A, which belongs to different universality class, the effective quantum field theory contains chiral 'relativistic' fermions, while the collective bosonic modes interact with these 'elementary particles' as gauge fields and gravity. All these fields emerge together with the Lorentz and gauge invariances and with elements of the general covariance from the fermionic Theory of Everything in eqn (3.2). The vacuum of the Standard Model belong to the same universality class, and the RQFT of the Standard Model is the corresponding effective theory.

The emergent phenomena do not depend much on the details of the Theory of Everything (Laughlin and Pines 2000) – in our case on the details of the pair potential $U(\mathbf{x}-\mathbf{y})$. Of course, the latter determines the universality class in which the system finds itself at low energy. But once the universality class is established, the physics remains robust to deformations of the pair potential. The details of $U(\mathbf{x}-\mathbf{y})$ influence only the 'fundamental' parameters of the effective theory ('speed of light', 'Planck' energy cut-off, etc.) but not the general structure of the theory. Within the effective theory the 'fundamental' parameters are considered as phenomenological.

3.1.4 *Fundamental constants of Theory of Everything*

The original number of fundamental parameters of the microscopic Theory of Everything is big: these are all the relevant Fourier components of the pair potential $U(r = |\mathbf{x} - \mathbf{y}|)$. However, one can properly approximate the shape of the potential. Typically the Lennard-Jones potential $U(r) = \epsilon_0 \left((r_0/r)^{12} - (r_0/r)^6 \right)$ is used, which simulates the hard-core repulsion of two atoms at small distances and their van der Waals attraction at large distances. This $U(r)$ contains only two parameters, the characteristic depth ϵ_0 of the potential well and the length r_0 which characterizes both the hard core of the atom and the dimension of the potential well.

Thus the microscopic Theory of Everything in a quantum liquid can be expressed in terms of four parameters: $\hbar$, ϵ_0, r_0 and the mass of the atom m. These 'fundamental constants' of the Theory of Everything determine the 'fundamental constants' of the descending effective theory at lower energy. On the other hand, we know that at least two of them, ϵ_0 and r_0, can be derived from the more fundamental Theory of Everything – atomic physics – whose 'fundamental constants' are $\hbar$, electric charge e and the mass of the electron m_e. In turn, e and m_e are determined by the higher-energy Theory of Everything – the Standard Model etc. Such a hierarchy of 'fundamental constants' indicates that the ultimate set of fundamental constants probably does not exist at all.

The Theory of Everything for liquid ^{3}He or ^{4}He does not contain a small parameter: the dimensionless quantity, which can be constructed from the four constants $r_0\sqrt{m\epsilon_0}/\hbar$, appears to be of order unity for ^{3}He and ^{4}He atoms. As a result the quantum liquids ^{3}He and ^{4}He are strongly correlated and strongly interacting systems. The distance between atoms in equilibrium liquids is determined by the competition of the attraction and the repulsing zero-point oscillations of atoms, and is thus also of order r_0. Zero-point oscillations of atoms prevent solidification: in equilibrium both systems are liquids. Solidification occurs when rather mild external pressure is applied.

Since there is no small parameter, it is a rather difficult task to derive the effective theory from first principles, though it is possible if one has enough computer time and memory. That is why it is instructive to consider the microscopic theory for some special model potentials $U(r)$ which contain a small parameter, but leads to the same universality class of effective theories in the low-energy limit. This allows us to solve the problem completely or perturbatively. In the case of the Bose–liquids the proper model is the Bogoliubov model (1947) of weakly interacting Bose gas; for the superfluid phases of ^{3}He it is the Bardeen–Cooper–Schrieffer (BCS) model.

Such models are very useful, since they simultaneously cover the low-energy edge of the effective theory and the Theory of Everything, i.e. high-energy 'trans-

Planckian' physics. In particular, this allows us to check the validity of different regularization schemes elaborated within the effective theory.

3.2 Weakly interacting Bose gas

3.2.1 *Model Hamiltonian*

In the Bogoliubov theory of the weakly interacting Bose gas the pair potential in eqn (3.2) is weak. As a result, in the vacuum state most particles are in the Bose–Einstein condensate, i.e. in the state with momentum $\mathbf{p} = 0$. The vacuum with Bose condensate is characterized by the scalar order paramater – the non-zero vacuum expectation value (vev) of the particle annihilation operator at $\mathbf{p} = 0$:

$$\langle a_{\mathbf{p}=0} \rangle = \sqrt{N_0} e^{i\Phi} \ , \ \left\langle a^\dagger_{\mathbf{p}=0} \right\rangle = \sqrt{N_0} e^{-i\Phi}. \tag{3.4}$$

Here $N_0 < N$ is the particle number in the Bose condensate, and Φ is the phase of the condensate. The vacuum state is not invariant under $U(1)_N$ global gauge rotations, and thus the vacuum states are degenerate: vacua with different Φ are distinguishable but have the same energy. Further we choose a particular vacuum state with $\Phi = 0$. Since the number of particles in the condensate is large, one can treat operators $a_{\mathbf{p}=0}$ and $a^\dagger_{\mathbf{p}=0}$ as classical fields, merely replacing them by their vev in the Hamiltonian.

If there is no interaction between the particles (an ideal Bose gas), the vacuum is completely represented by the Bose condensate particles, $N_0 = N$. The interaction pushes some fraction of particles from the $\mathbf{p} = 0$ state. If the interaction is small, the fraction of the non-condensate particles in the vacuum is also small, and they have small momenta $\mathbf{p}$. As a result, only the zero Fourier component of the pair potential is relevant, and the potential can be approximated by a δ-function, $U(\mathbf{r}) = U\delta(\mathbf{r})$. The Theory of Everything in eqn (3.2) then acquires the following form:

$$\mathcal{H} - \mu\mathcal{N} = -\mu N_0 + \frac{N_0^2 U}{2V} \tag{3.5}$$

$$+ \sum_{\mathbf{p}\neq 0} \left(\frac{p^2}{2m} - \mu \right) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{3.6}$$

$$+ \frac{N_0 U}{2V} \sum_{\mathbf{p}\neq 0} \left(2a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + 2a^\dagger_{-\mathbf{p}} a_{-\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} + a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} \right) \ . \tag{3.7}$$

We ignore quantum fluctuations of the operator a_0 considering it as a c-number: $N_0 = a_0^\dagger a_0 = a_0 a_0^\dagger = a_0^\dagger a_0^\dagger = a_0 a_0$. Note that the last two terms in eqn (3.7) do not conserve particle number: this is the manifestation of the broken $U(1)_N$ symmetry in the vacuum.

Minimization of the main part of the energy in eqn (3.5) over N_0 gives $UN_0/V = \mu$ and one obtains

$$\mathcal{H} - \mu\mathcal{N} = -\frac{\mu^2}{2U} V + \sum_{\mathbf{p}\neq 0} \mathcal{H}_{\mathbf{p}} \ , \tag{3.8}$$