General information

The exercise sessions will be held as blackboard sessions, where the participants will present their solutions to the group. As such, the problems should be set up and solved before the session. The focus of the exercises lies on analyzing and discussing the task at hand together with the group: thus, a perfect solution is not required to be awarded points. A point will be awarded for each question, and a person will be chosen to present their solution from the pool. *There are useful formulas at the bottom of this document!*

Exercise 1.

Energy confinement times and ignition

A central figure of merit within the field of fusion technology is the fusion triple product. The fusion triple product relates the fuel density, n_{DT}, fuel temperature, T, and energy confinement time, τ_E, to successful, self-sustained fusion burn. The triple product for deuterium-tritium fusion is

$$n_{DT} T \tau_E \geq 10^{22} \text{ keV s m}^{-3}.$$

Calculate the required confinement times for ignition in the following confinement concepts:

(a) **Magnetic confinement:** a tokamak with fuel density $n \sim 10^{20}$ m^{-3} and temperature $T \sim 10$ keV. Compare the value to the one foreseen for ITER.

(b) **Inertial confinement:** a laser fusion device with $n \sim 10^{32}$ m^{-3} and fuel temperature of $T \sim 10$ keV. Compare the obtained value to the ratio of the fuel sound speed ($c_s \sim 10^5$ m/s) to the fuel capsule radius ($R \sim 100$ μm).

Solution 1.

(a) $\tau_{\mathrm{E}} \sim 10$ s, which is close to the ITER reference value of 8 s.

(b) $\tau \sim 10^{-11}$ s. The estimated inertial confinement time (ratio of fuel capsule radius to ion sound speed) is about 10^{-9} s.

Exercise 2.

Power densities and wall power loads in different fusion reactors

Estimate the power densities and wall power loads in the following fusion reactors:

(a) **Tokamak:** A tokamak operating with a 50-50 D-T fuel mix with major radius $R = 5.5$ m and minor radius $a = 1.8$ m. Assume flat temperature and density profiles (= constant temperature and density) with $n_D = n_T = 10^{20}$ m^{-3} and $T_e = T_i = 10$ keV. Use the low temperature approximation from the "Fusion principles" lecture notes to determine the mean reaction rate.

(b) **Z-pinch:** A Z-pinch with $R = 10$ cm operating with a 50-50 DT fuel mix. The fusion reactions occur in a narrow rope with a diameter of 40 μm. The DT density in the rope is $n_{DT} = 4.5 \times 10^{28}$ m^{-3}. The burn-up of the fuel is 30% and the pulse rate, f, of the device is 1 Hz. See figure *Fig.* 1 for illustration of the assumed geometry.

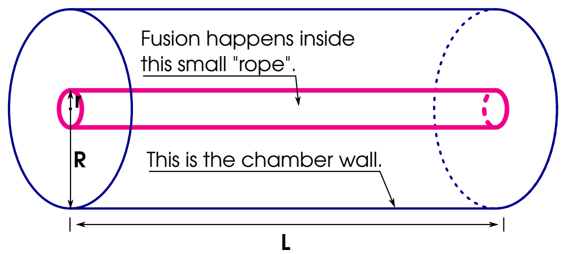

Figure 1: An illustration of the assumed z-pinch geometry in exercise 2b).

Solution 2.

(a) The power density of the reactor is

$$P_f = n_D n_T \langle \sigma v \rangle E_{DT} \approx 1.3 \times 10^{19} \ \ \mathrm{MeVm^{-3}s^{-1}} = 2.1 \ \ \mathrm{MWm^{-3}},$$

where the low temperature approximation of the mean reaction rate has been used: $\langle \sigma v \rangle_{D-T} \approx 7.58 \times 10^{-23} \ \ \mathrm{m^3s^{-1}}$.

The wall power loads of the reactor are given by

$$q_{wall} = \frac{P_f V_{tokamak}}{A_{tokamak}} = P_f \frac{\cdot \pi a^2 \cdot 2\pi R}{2\pi a \cdot 2\pi R} = P_f \frac{a}{2} \approx 1.9 \ \ \mathrm{MWm^{-2}}$$

(b) We assume the pinch to be of infinite length, and the fusion reactions to occur in the narrow rope in the center of the device. The power density is thus

$$P_{f,rope} = \frac{n_{DT}}{2} E_{DT} \eta \cdot f_{pulse} \approx 1.19 \times 10^{29} \ \ \mathrm{MeV\ s^{-1}\ m^{-3}} = 1.9 \times 10^{16} \ \ \mathrm{W\ m^{-3}},$$

where the factor $1/2$ comes from the fuel density considering both deuterium and tritium particles, η is the burn-up and f_{pulse} is the pulse frequency. The total power produced per meter of the reactor is, thus,

$$P_{rope}/l = P_f * A_{rope} = P_f \cdot \pi \left(\frac{d}{2}\right)^2 \approx 23.9 \ \ \mathrm{MW/m}.$$

The geometric properties, per meter of length, of the machine are

$$\begin{aligned} V_{Z-pinch}/l =& \pi R^2 \approx 0.0314 \ \mathrm{m^3/m} \\ A_{Z-pinch}/l =& 2\pi R \approx 0.628 \ \ \mathrm{m^2/m} \end{aligned}$$

The total Z-pinch power density and wall power flux thus become

$$P_{f,Z-pinch} = \frac{P_{rope}/l}{V_{Z-pinch}/l} \approx 761 \ \ \text{MW m}^{-3}$$
$$q_{f,Z-pinch\ wall} = \frac{P_{rope}/l}{A_{Z-pinch}/l} \approx 38 \ \ \text{MW m}^{-2}$$

If we assume the Z-pinch to be of finite length, we will have to consider the area of the cylinder ends, thus the wall power flux becomes variable, increasing with increasing machine length, with the above wall load as its maximum. If we e.g. use a 10 cm long machine the wall power load becomes

$$q_{wall,10\ \ \text{cm}} = \frac{P_{f,Z-pinch} V_{Z-pinch,10\ \text{cm}}}{A_{wall,10\ \text{cm}}} = \frac{P_{f,Z-pinch} 0.1\ \text{m}\pi R^2}{2\pi R \cdot 0.1\ \text{m} + 2 \times 2\pi R^2} \approx 19\ \text{MW m}^{-2}$$

Exercise 3.
Sun and stars

In the sun, energy is released primarily through the following chain of reactions:

$$\text{p} + \text{p} \longrightarrow \text{D} + \text{e}^+ + \nu_\text{e} + 1.44\,\text{MeV}$$
$$\text{D} + \text{p} \longrightarrow {}^3\text{He} + \gamma + 5.49\,\text{MeV}$$
$${}^3\text{He} + {}^3\text{He} \longrightarrow {}^4\text{He} + 2\,\text{p} + \gamma + 12.89\,\text{MeV}$$

(a) Show that the energy released per proton fused in this reaction chain is approximately 6.68 MeV. What happens to the positrons and the neutrinos?

(b) Find reasonable estimates for the power output and mass of the sun and use them to calculate the reaction rate per proton per second in the sun. Assume that the power is produced in the core, which contains 10% of the total mass. How long does the average proton have to wait before it fuses in the sun?

Solution 3.

(a) We can find the total energy released by first multiplying the second reaction equation by two and adding it to the third equation:

$$2 \cdot (\text{D} + \text{p} \longrightarrow {}^3\text{He} + \gamma + 5.49\,\text{MeV})$$
$${}^3\text{He} + {}^3\text{He} \longrightarrow {}^4\text{He} + 2\,\text{p} + \gamma + 12.89\,\text{MeV}$$

$$2\,\text{D} + 2\,\text{p} + 2\,{}^3\text{He} \longrightarrow 2\,{}^3\text{He} + {}^4\text{He} + 2\,\text{p} + 3\,\gamma + 23.87\,\text{MeV}$$

or, after cancelling terms,

$$2\,\mathrm{D} + \;\longrightarrow\; {}^4\mathrm{He} + 3\,\gamma + 23.87\,\mathrm{MeV}$$

Next, multiply and add the first equation to the above one:

$$\begin{aligned} 2\cdot(\mathrm{p} + \mathrm{p} &\longrightarrow \mathrm{D} + \mathrm{e}^+ + \nu_\mathrm{e} + 1.44\,\mathrm{MeV}) \\ 2\,\mathrm{D} + \; &\longrightarrow {}^4\mathrm{He} + 3\,\gamma + 23.87\,\mathrm{MeV} \end{aligned}$$

$$2\,\mathrm{D} + 4\,\mathrm{p} \longrightarrow 2\,\mathrm{D} + 2\,\mathrm{e}^+ + 2\,\nu_\mathrm{e} + {}^4\mathrm{He} + 3\,\gamma + 26.75\,\mathrm{MeV}$$

Cancelling terms gives the total reaction:

$$4\,\mathrm{p} \longrightarrow {}^4\mathrm{He} + 2\,\mathrm{e}^+ + 2\,\nu_\mathrm{e} + 3\,\gamma + 26.75\,\mathrm{MeV}$$

The energy released per proton is thus $26.75\,\mathrm{MeV}/4 = 6.6875\,\mathrm{MeV}$. The two positrons are annihilated in collisions with electrons in the core. Each annihilation event releases $2 \cdot 0.511\,\mathrm{MeV} = 1.02\,\mathrm{MeV}$ of energy, which has already been accounted for in the reaction equations. The neutrinos most likely escape the sun without depositing their energy there. In part b, we have neglected any complication arising from the variable amount of energy per each reaction that is lost to the neutrinos (on average 0.26 MeV per neutrino).

(b) The power output of the sun is approximately $P_S = 3.8 \times 10^{26}\,\mathrm{W}$ and its mass is roughly $M_S = 1.989 \times 10^{30}\,\mathrm{kg}$. The energy release per proton fused is $E_p = 6.68 \times 10^6\,\mathrm{eV} \cdot 1.602 \times 10^{-19}\,\mathrm{J\,eV^{-1}} = 1.07 \times 10^{-12}\,\mathrm{J}$. We can now calculate the number of protons in the core:

$$N_{p,c} = \frac{M_S}{10 \cdot m_p}$$

The number of reactions per second (f) per proton in the core is

$$\frac{f}{N_{p,c}} = \frac{P_S}{E_p N_{p,c}} = \frac{10 \cdot P_S m_p}{E_p M_S}$$

Plugging in values

$$\frac{10 \cdot P_S m_p}{E_p M_S} = \frac{10 \cdot 3.8 \times 10^{26}\,\mathrm{W} \cdot 1.67 \times 10^{-27}\,\mathrm{kg}}{1.07 \times 10^{-12}\,\mathrm{J} \cdot 1.989 \times 10^{30}\,\mathrm{kg}} = 2.98 \times 10^{-18}\,\mathrm{s}^{-1}$$

The estimated average wait time is thus $1/2.98 \times 10^{-18}\,\mathrm{s}^{-1} = 3.35 \times 10^{17}\,\mathrm{s}$, or approximately 10 billion years (the final figure could also be lower by a factor of two, depending on the way you estimated it).

Exercise 4.
Inertial confinement

In a laser fusion device, a spherical symmetric fusion target is compressed to 1000 times the initial density: $\rho = 1000\rho_0$, or $R = 0.1R_0$. The initial density of the fuel pellet, consisting of cooled, solid D-T, is $\rho_0 = 0.23$ g/cm^3 and the radius is $R_0 = 0.2$ cm.

(a) What is the minimum energy of the laser pulse E_{min} required, if a temperature of 5 keV is required for the ignition? Assume that the fuel compresses adiabatically, i.e. $pV^\gamma =$ const in the process, with $\gamma = 5/3$.

(b) What is the fusion gain factor (the ratio of fusion energy released to the energy needed)? The burn-up B, the fraction of the fuel that fuses, is needed, and can be calculated according to:

$$B = \frac{\rho R}{\rho R + 6\frac{\text{g}}{\text{cm}^2}},$$

where ρ is the mass density of the fuel.

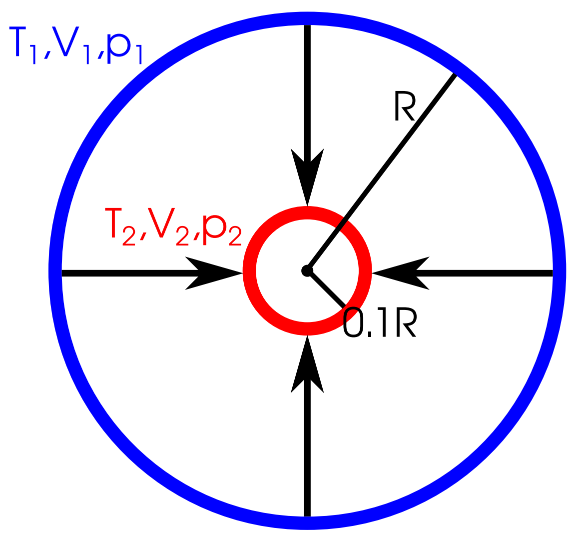

Figure 2: Illustration of the compression in laser fusion.

Solution 4.

(a) Assuming adiabatic compression, for which

$$pV^\gamma = \text{constant},$$

where p is the pressure, V the volume, and γ the ratio of the specific heats:

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f},$$

where f is the numbers of degrees of freedom. We approximate the deuterium-tritium mixture as a monatomic gas, where the degrees of freedom are the spatial directions in three-dimensional space, $f = 3 \to \gamma = 5/3$.

The energy required to compress the monatomic gas adiabatically is found by

$$W = \int_{V_1}^{V_2} p dV = \int_{V_1}^{V_2} \frac{p_2 V_2^\gamma}{V^\gamma} dV = p_2 V_2^\gamma \frac{V_2^{1-\gamma} - V_1^{1-\gamma}}{1-\gamma}$$
$$= \frac{p_2 V_2}{1-\gamma}\left(1 - \frac{V_2^{\gamma-1}}{V_1^{\gamma-1}}\right) = \frac{N_2 T_2}{1-\gamma}\left(1 - \frac{V_2^{\gamma-1}}{V_1^{\gamma-1}}\right).$$

The particle density in the initial state is $n_0 \approx \rho_0/(2.5 \cdot m_p)$. Therefore, the total amount of particles in the fuel pellet is $N_0 = N_2 = V_{\text{pellet, 0}} \cdot n_0 = \frac{4}{3}\pi r^3 n_0 = 1.843 \cdot 10^{21}$.

Therefore, the energy needed to adiabatically compress the fuel to these conditions is about 2.19 MJ.

(b) The mass density times the radius of the fuel is $\rho R = 1000\rho_0 0.1 R_0 = 4.6$ g/cm^2, yielding a burn-up of $B \approx 0.43$. Therefore, the fusion energy production can be calculated: $W_F = 0.5 n E_{DT} V B = 0.5 N_2 E_{DT} B \approx 1.12$ GJ, resulting in a gain factor of about 511.

Constants:
1 eV $= 1.602 \times 10^{-19}$ J
$m_p = 1.673 \times 10^{-27}$ kg
$m_n = 1.675 \times 10^{-27}$ kg
$N_A = 6.022 \times 10^{23}$ mol^{-1}
$k_B = 1.381 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Power equations assuming pure hydrogenic (Z=1) plasma:

Fusion power density: $P_f = \alpha n_i n_j \langle \sigma v \rangle E_f$,

where E_f represents the produced energy per a fusion reaction, n_i and n_j are the fuel isotope densities, n_e the electron density, T_S the surface temperature of the black body, and A the surface area of the black body. The α parameter in the fusion power density equation is 1 for D-T fusion, and 1/2 for D-D fusion. This parameter arises due to the fact that when calculating the fusion cross-section integral ($\langle \sigma v \rangle$) for like particle collisions (D-D), every collision is counted twice. This should not be confused with the 1/4-factor that arises in the D-T fusion cross-section with 50-50 % fuel mixture due to $n_D = n_T = n_e/2 \rightarrow n_D \times n_T = (1/4) \times n_e$. More information can be found in e.g. [1].

References:

[1] J. P. Freidberg, Plasma Physics and Fusion Energy, Cambridge University Press, 2007, p.44