## (9) Versions of the Chain Rule (for differentiating a composition of functions)

CASE 1:

$$f \circ g : \mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$x \longmapsto g(x) \longmapsto f(g(x))$$

$$\boxed{\frac{d}{dx} f(g(x)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Ex:

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R} \qquad (f \circ g)(x) = \sin x^2$$
$$x \longmapsto x^2 \longmapsto \sin x^2$$

$$(f \circ g)'(x) = \cos x^2 \cdot 2x$$

CASE 2:

$$f \circ g : \mathbb{R}^2 \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$(x,y) \longmapsto g(x,y) \longmapsto f(g(x,y))$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} f(g(x,y)) &= f'(g(x,y)) \cdot \frac{\partial}{\partial x} g(x,y) \\ \frac{\partial}{\partial y} f(g(x,y)) &= f'(g(x,y)) \cdot \frac{\partial}{\partial y} g(x,y) \end{aligned}}$$

Ex:

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$(x,y) \longmapsto x^2+y^2 \longmapsto \sin(x^2+y^2)$$

$$\frac{\partial}{\partial x} f(g(x,y)) = \cos(x^2+y^2) \cdot 2x$$

$$\frac{\partial}{\partial y} f(g(x,y)) = \cos(x^2+y^2) \cdot 2y$$

CASE 3: $\varepsilon: \mathbb{R} \xrightarrow{u,v} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$t \longmapsto (u(t), v(t)) \longmapsto f(u(t), v(t))$

$\varepsilon(t) = f(u(t), v(t))$

$$\boxed{\varepsilon'(t) = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}} \quad (\text{III})$$

Ex: $\mathbb{R} \xrightarrow{u,v} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ $\quad \boxed{f(u,v) = u \cdot v^2}$

$t \longmapsto (\underbrace{\cos t}_{u(t)}, \underbrace{t^3}_{v(t)}) \longmapsto \cos t \cdot t^6$

(I) By direct substitution. $\varepsilon(t) = f(u(t), v(t)) = \cos t \cdot t^6$

$\varepsilon'(t) = -\sin t \cdot t^6 + \cos t \cdot 6t^5$

(II) By Chain rule (III)

$\frac{\partial f}{\partial u} = v^2 = t^6$ , $\frac{du}{dt} = -\sin t$

$\frac{\partial f}{\partial v} = 2uv = 2 \cdot \cos t \cdot t^3$ ; $\frac{dv}{dt} = 3t^2$

$\varepsilon'(t) = t^6 \cdot (-\sin t) + 2t^3 \cdot \cos t \cdot 3t^2 =$

$= -\sin t \cdot t^6 + \cos t \cdot 6t^5$

CASE 4: $\varepsilon: \mathbb{R}^2 \xrightarrow{u,v} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$(s,t) \longmapsto (u(s,t), v(s,t)) \longmapsto f(u(s,t), v(s,t))$

$\varepsilon(s,t) = f(u(s,t), v(s,t))$

$$\boxed{\begin{aligned} \frac{\partial \varepsilon}{\partial s} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial s} \\ \frac{\partial \varepsilon}{\partial t} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t} \end{aligned}} \quad (\text{IV})$$

Ex:

$$\varepsilon: \mathbb{R}^2 \xrightarrow{u,\,v} \mathbb{R}^2 \xrightarrow{f} \mathbb{R} \qquad \boxed{f(u,v)=\sin(u^2v)}$$

$$(s,t) \longmapsto (st^2,\ s^2+\tfrac{1}{t}) \longmapsto \sin\left(s^2t^4\left(s^2+\tfrac{1}{t}\right)\right)$$

$$\underset{u(s,t)}{\overset{\shortparallel}{\phantom{x}}} \qquad \underset{v(s,t)}{\overset{\shortparallel}{\phantom{x}}}$$

(I) By direct substitution and case 2:

$$\varepsilon(s,t) = \sin\left(s^2t^4\left(s^2+\tfrac{1}{t}\right)\right) = \sin\left(s^4t^4+s^2t^3\right)$$

- $\dfrac{\partial \varepsilon}{\partial s} = \cos\left(s^4t^4+s^2t^3\right)\cdot\left(4s^3t^4+2st^3\right)$
- $\dfrac{\partial \varepsilon}{\partial t} = \cos\left(s^4t^4+s^2t^3\right)\cdot\left(4s^4t^3+3s^2t^2\right)$

(II) By Chain rule (IV) $\varepsilon(s,t)=f(u(s,t),v(s,t))$

$$\frac{\partial f}{\partial u} = \cos(u^2v)\cdot 2uv \ ; \quad \frac{\partial u}{\partial s} = t^2 \ ; \quad \frac{\partial u}{\partial t} = 2st$$

$$\frac{\partial f}{\partial v} = \cos(u^2v)\cdot u^2 \ ; \quad \frac{\partial v}{\partial s} = 2s \ ; \quad \frac{\partial v}{\partial t} = -\frac{1}{t^2}$$

- $$\frac{\partial \varepsilon}{\partial s} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial s} = \underbrace{\cos(s^4t^4+s^2t^3)\cdot 2\cdot st^2\cdot\left(s^2+\tfrac{1}{t}\right)}\cdot t^2 + \underbrace{\cos(s^4t^4+s^2t^3)\cdot s^2t^4}\cdot 2s =$$
$$= \cos\left(s^4t^4+s^2t^3\right)\left(4s^3t^4+2st^3\right)$$

- $$\frac{\partial \varepsilon}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t} = \underbrace{\cos(s^4t^4+s^2t^3)\cdot 2\cdot st^2\cdot\left(s^2+\tfrac{1}{t}\right)}\cdot 2st + \underbrace{\cos(s^4t^4+s^2t^3)\cdot s^2t^4}\cdot\left(-\tfrac{1}{t^2}\right)$$
$$= \cos\left(s^4t^4+s^2t^3\right)\cdot\left(4s^4t^3+3s^2t^2\right)$$