General information

The exercise sessions will be held as blackboard sessions, where the participants will present their solutions to the group. As such, the problems should be set up and solved before the session. The focus of the exercises lies on analyzing and discussing the task at hand together with the group: thus, a perfect solution is not required to be awarded points. A point will be awarded for each question, and a person will be chosen to present their solution from the pool.

Exercise 1.
Magnetic and electric fields inside a tokamak

(a) The toroidal field in a tokamak is induced by running a steady current I_{tc} in each of the N_{tc} toroidal coils. Use Ampere's law to calculate how the strength of the toroidal magnetic field inside a torus varies as a function of s, the distance from the symmetry axis of the torus. What is the field strength outside of the toroidal coils? Assume that the toroidal coils are very close together.

(b) The plasma current in a tokamak is driven inductively by running a time varying current $I_{cs}(t)$ in the central solenoid. Use Faraday's law to calculate the induced toroidal electric field as a function of distance from the symmetry axis of the torus, assuming that the central solenoid consists of N_{cs} coils and the current in each varies as $\frac{\partial I(t)}{\partial t}$. Assume that the solenoid is very long.

Solution 1.

(a) It is assumed that the magnetic field is purely toroidal (Fig. 1). For a detailed proof of this fact, see "Introduction to Electrodynamics", Griffiths, pp. 240-241, 4th edition.

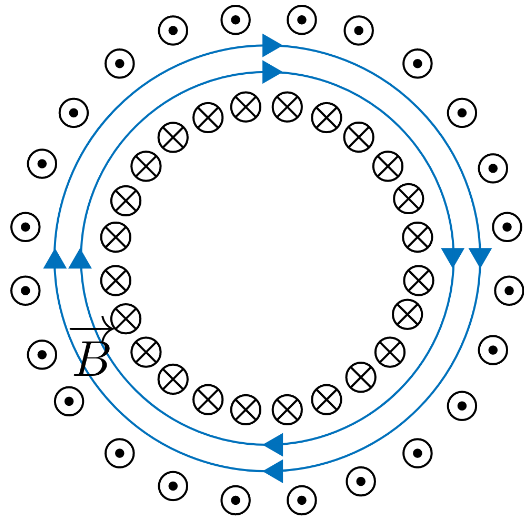


Figure 1: Current running in the toroidal field coils and the toroidal magnetic field in a tokamak. Figure credit J. Hedberg, 2018.

Ampere's law in integral form reads

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{enc}}, \tag{1}$$

where I_{enc} inside an Amperian loop (the path around which the integration is performed). If we draw the Amperian loop inside the toroidal coils (Fig. 2), the enclosed current is the number of coils times the current running in each coil, $I_{\mathrm{enc}} = N_{\mathrm{tc}} I_{\mathrm{tc}}$.

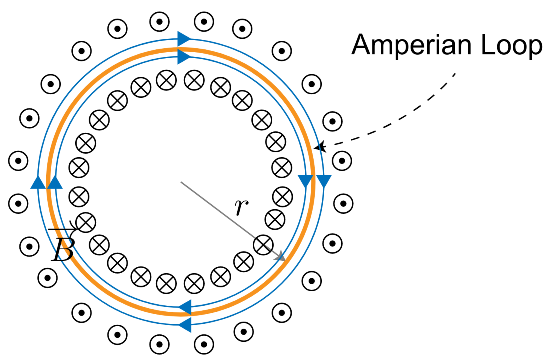


Figure 2: Amperian loop (in orange) inside the toroidal field coils. Figure credit J. Hedberg, 2018.

An Amperian loop at a distance s from the symmetry axis gives

$$B \cdot 2\pi s = \mu_0 N_{\mathrm{tc}} I_{\mathrm{tc}}, \tag{2}$$

where we have used the fact that B is a constant along the path of integration and can be taken outside the integral. Thus, the magnetic field inside the toroidal coils is

$$B = \frac{\mu_0 N_{\text{tc}} I_{\text{tc}}}{2\pi s}, \tag{3}$$

Drawing the Amperian loop outside the toroidal field coils results in net zero enclosed current. Therefore, the field outside of the coils is zero.

(b) The magnetic field inside an infinitely long solenoid is given by:

$$\mathbf{B} = \mu_0 n I \hat{\mathbf{z}}, \tag{4}$$

where $\hat{\mathbf{z}}$ is a unit vector pointing along the symmetry axis of the solenoid in a direction given by the right hand rule. Outside the solenoid the magnetic field is zero. For a derivation, see "Introduction to Electrodynamics", Griffiths, pp. 237-239, 4th edition. In our exercise, $n = N_{cs}/L$ is the number of coils per units length (L is the length of the central solenoid). The magnetic field varies as

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\mu_0 N_{cs}}{L} \frac{\partial I}{\partial t} \hat{\mathbf{z}}. \tag{5}$$

Faraday's law states that

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a}\,, \tag{6}$$

where the LHS integral is performed around an Amperian loop. The RHS integral is performed over the area enclosed inside the Amperian loop. The Amperian loop is set to be a circle of radius s around the symmetry axis of the torus. In this case, the only magnetic flux enclosed within the Amperian loop is that inside the solenoid (of radius a). Since E is constant along the path of integration, and B is spatially constant, E and B can be taken outside of their respective integrals, giving

$$E \cdot 2\pi s = -\frac{\mu_0 N_{cs}}{L} \frac{\partial I}{\partial t} \cdot \pi a^2, \tag{7}$$

which simplifies to

$$E = -\mu_0 \frac{a^2 N_{cs}}{2sL} \frac{\partial I}{\partial t}. \tag{8}$$

Exercise 2.
Plasma confinement in tokamaks

In order to understand the plasma confinement in tokamaks, one must consider the loss- and transport phenomenon of heat and particles in the plasma. Classical and neoclassical predictions for radial transport in tokamaks will be investigated. These theoretical findings will be compared to experimentally observed values. Despite the many calculations and the platitude of information, the mathematics are straightforward, so do not lose hope.

(a) The experimentally observed confinement time in tokamaks is $\tau_E \sim 1$ s, while the minor radii in reactor scale tokamaks are approximately $a \sim 1$ m. Based on these ballpark values, estimate the radial transport (diffusion) coefficient for tokamaks, assuming $D \sim r^2/\tau$.

(b) Using the classical concept of collisional diffusion, the diffusion coefficient can be expressed as $D \sim (\text{step size})^2 \times (\text{collisionfrequency})$. Thus, the classical estimate for the radial diffusion coefficient is given by

$$D_{\text{classic}} \sim (r_{L,e})^2 \nu_{ei},$$

where $r_{L,e}$ is the electron gyroradius and ν_{ei} is the electron-ion collision frequency. The classical collisional thermal conductivity coefficients are

$$\chi_{s,\text{classic}} \sim (r_{L,s})^2 \nu_{ss}$$

where the subscript s refers to electrons and ions ($s \in \{i, e\}$), $r_{L,s}$ is the ion/electron gyro radius, while ν_{ss} are the ion-ion/electron-electron self-collision frequencies. Now, calculate the resulting values for D_{classic}, $\chi_{i,\text{classic}}$, and $\chi_{e,\text{classic}}$ assuming $\nu_{ei} \sim 4.6 \times 10^3\ \text{s}^{-1} \sim \nu_{ee}$, $\nu_{ii} \sim (m_e/m_i)^{1/2}\nu_{ei}$, $T_e = T_i \sim 5$ keV, and $B_0 \sim 5$ T.

(c) Due to the geometry of the tokamak, the toroidal field coils are denser towards the axis of symmetry, resulting in a stronger magnetic field closer to the axis of symmetry. Thus, the side of the plasma close to the axis of symmetry is referred to as the high field side (HFS), and the outer plasma as the low field side (LFS). This inhomogeneity in the toroidal magnetic field further enhances the transport in the device, due to cross-field drifts. These drifts effectively increase the diffusion step size, and the resulting transport, from the classical step size, and the resulting transport is referred to as neoclassical transport.

The radial gradient in the magnetic field also results in a phenomenon called particle trapping, in which a fraction of the particles are reflected before reaching the HFS plasma. Most of these trapped particles complete banana shaped orbits and are, naturally, called banana trapped particles, while the non-trapped particles are referred to as passing particles. Figure 3 should help, but watching an animation is even better, e.g. `https://www.youtube.com/watch?v=XUhNium3VEo`.

Calculate the diffusion coefficient estimates for both trapped and passing particles, using the neoclassical transport coefficient approximation for passing particles:

$$D^p_{\text{neoclassic}} \sim 4q^2 D_{\text{classic}}$$
$$\chi^p_{s,\text{neoclassic}} \sim q^2 \chi_{s,\text{classic}}$$

where q is the safety factor, a quantity related to the windedness of the magnetic field. The neoclassical transport coefficient for trapped particles can then be estimated as

$$D^t_{\text{neoclassic}} \sim 2.2q^2 (R_0/r)^{3/2} D_{\text{classic}}$$
$$\chi^t_{e,\text{neoclassic}} \sim 0.89q^2 (R_0/r)^{3/2} \chi_{e,\text{classic}}$$
$$\chi^t_{i,\text{neoclassic}} \sim 0.68q^2 (R_0/r)^{3/2} \chi_{i,\text{classic}}$$

Calculate the neoclassical estimates for the radial transport coefficients assuming $R_0 \sim 3$ m, $r \sim 1$ m, and $q \sim 3$.

(d) What conclusions can you make regarding classical and neoclassical collisional diffusion with regards the estimated cross-field transport calculated in a)?

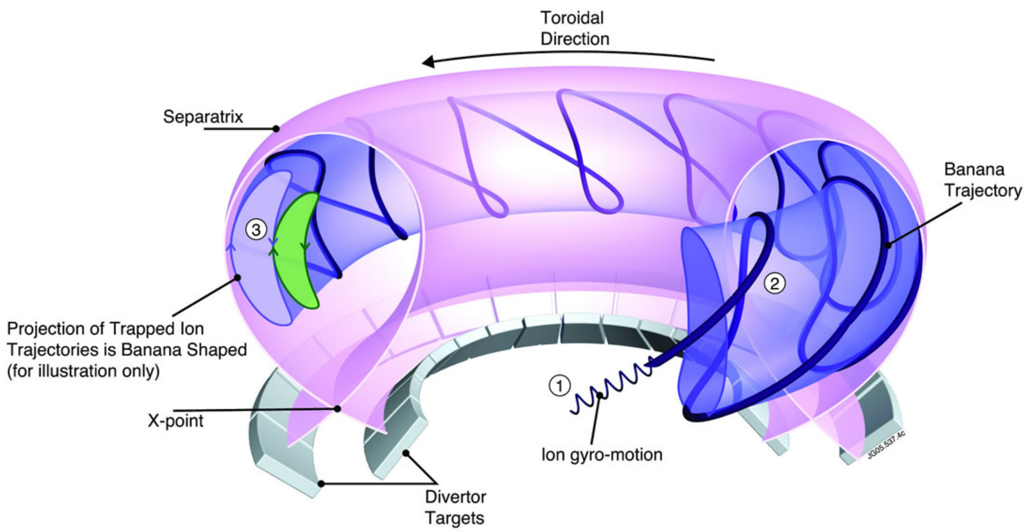


Figure 3: Illustration of trapped particle orbits in a tokamak. The phenomenon is not limited just to fusion devices but occurs also in e.g. Earth's magnetosphere.

Solution 2.

(a) Energy flux through the plasma core boundary of a toroidal configuration can be estimated from the confinement time τ_E, the energy in the plasma W, and the plasma surface area A:

$$J = \frac{W}{A\tau_E} = \frac{3k_BTn_eV}{A\tau} = \frac{3k_BTn_e 2\pi R \cdot \pi r^2}{2\pi R \cdot 2\pi r\tau} = \frac{3k_BTn_e \cdot r}{2\tau}.$$

Fick's law relates the the energy flux to the diffusion coefficient, assuming constant J:

$$J = D\nabla\frac{W}{V} = \frac{3k_BTn_e}{r} = \frac{3k_BTn_e \cdot r}{2\tau} \rightarrow D = \frac{r^2}{2\tau_E} \sim \frac{r^2}{\tau_E}$$

giving the approximation

$$D_{classical} \approx 1 \text{ m}^2\text{s}^{-1}$$

(b) The Larmor radius is the radius of the circular path of motion a charged particle in a magnetic field follows. By equating the centripetal acceleration of a particle j to Lorentz force:

$$m_j\frac{v_\perp^2}{r_L} = qv_\perp B \rightarrow r_L = \frac{m_jv_\perp}{|q|B},$$

where the velocity can be calculated by assuming thermal particles, i.e. the particles have thermal velocity

$$v \approx \sqrt{\frac{2T}{m}}.$$

Thus

$$\begin{aligned} r_{L,e} &= \frac{\sqrt{2T_e m_e}}{eB} \approx 48\ \mu\text{m} \\ r_{L,i} &= \frac{\sqrt{2T_i m_i}}{eB} \approx 2\ \text{mm} \approx \frac{m_i}{m_e} r_{L,e}, \end{aligned}$$

yielding

$$\begin{aligned} D_{\text{classic}} &\sim (r_{L,e})^2 \nu_{ei} \approx 1.2 \times 10^{-5}\ \text{m}^2\text{s}^{-1} \\ \chi_{i,\text{classic}} &\sim (r_{L,i})^2 \nu_{ii} \approx 4.3 \times 10^{-4}\ \text{m}^2\text{s}^{-1} \\ \chi_{e,\text{classic}} &\sim (r_{L,e})^2 \nu_{ee} \approx 1.2 \times 10^{-5}\ \text{m}^2\text{s}^{-1} \end{aligned}$$

The classical diffusion/thermal conduction coefficients are 3 to 5 orders of magnitude smaller than the experimental estimate, indicating some other mechanism must be driving the cross-field transport in the tokamak.

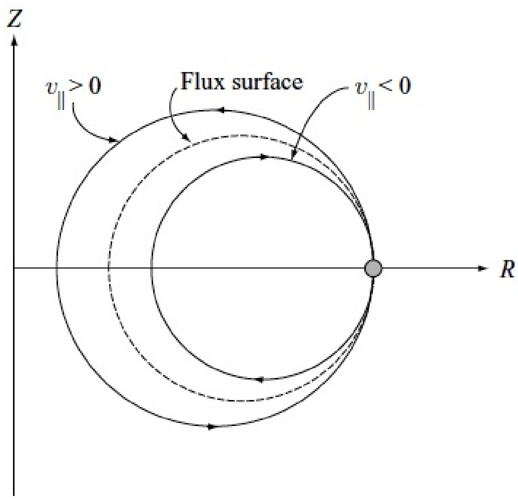
(c) The resulting neoclassical collisional diffusion/thermal conduction coefficients for passing particles are:

$$\begin{aligned} D_{\text{neo-classic, passing}} &\sim 4q^2 D_{\text{classic}} \sim 3.6 \times 10^{-4}\ \text{m}^2/\text{s} \\ \chi_{\text{i, neo-classic, passing}} &\sim q^2 \chi_{\text{i, classic}} \sim 3.9 \times 10^{-3}\ \text{m}^2/\text{s} \\ \chi_{\text{e, neo-classic, passing}} &\sim q^2 \chi_{\text{e, classic}} \sim 1.1 \times 10^{-4}\ \text{m}^2/\text{s} \end{aligned}$$

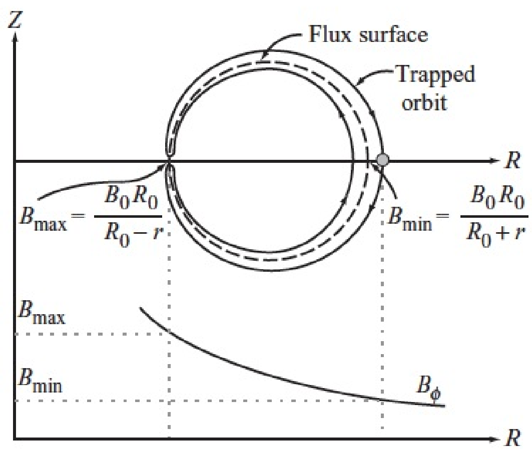
The coefficients for trapped particles are:

$$\begin{aligned} D_{\text{neo-classic, trapped}} &\sim 2.2q^2 (R_0/r)^{3/2} D_{\text{classic}} \sim 1.0 \times 10^{-3}\ \text{m}^2/\text{s} \\ \chi_{\text{i, neo-classic, trapped}} &\sim 0.68q^2 (R_0/r)^{3/2} \chi_{\text{i, classic}} \sim 1.3 \times 10^{-2}\ \text{m}^2/\text{s} \\ \chi_{\text{e, neo-classic, trapped}} &\sim 0.89q^2 (R_0/r)^{3/2} \chi_{\text{e, classic}} \sim 5 \times 10^{-4}\ \text{m}^2/\text{s} \end{aligned}$$

Therefore, it is generally found that for the classical estimates $\chi_{\text{e}} < D < \chi_{\text{i}}$. Furthermore, it is observed that $D_{\text{classic}} < D_{\text{passing}} < D_{\text{trapped}}$.



(a) Passing particles orbit



(b) Trapped particles orbits

(d) Even if the neoclassical transport is included, the experimental transport exceeds the calculated one by a factor of 100. The experimentally observed cross-field transport is driven by the micro turbulence of the plasma and cannot be explained by the simple classical collisional model presented here.

Exercise 3.
Trapped particles

Consider a particle starting at the outer mid-plane (LFS) of the poloidal cross-section of the torus with velocity $\mathbf{v}_0 = v_{\parallel,0}\mathbf{b} + \mathbf{v}_{\perp,0}$, where $\parallel$ stands for parallel and $\perp$ for perpendicular relative to the magnetic field direction $\mathbf{b}$. The kinetic energy of the particle can similarly be divided into perpendicular and parallel components. Ref. [1] might prove helpful in these calculations.

(a) Calculate the particle trapping condition for the pitch $\left(\frac{v_{\parallel,0}}{v_0} < \sqrt{1 - \frac{B_{\min}}{B_{\max}}}\right)$. Consider the extreme cases of the particle being located at at the LFS (minimum B) and bouncing HFS (maximum B), and use the conservation of energy and magnetic moment $\left(\mu = \frac{mv_\perp^2}{2B}\right)$ and the fact that the parallel velocity is momentarily zero at the bounce.

(b) Show that this condition is equivalent to the one given in the lecture slides $\left(\frac{v_{\parallel,0}}{v_0} < \sqrt{2\frac{r}{R_0}}\right)$ when assuming $R_0 >> r$ and $B(x) = \frac{B_0 R_0}{R_0 + x}$ with $-r \leq x \leq r$. B_0 is the magnetic field at the magnetic axis (in the middle of the cross section of the torus).

(c) Assuming Maxwellian velocity distribution, integrate over the trapped cone in the velocity space in spherical coordinates to obtain the fraction of trapped particles, i.e. calculate

$$f_t = \frac{1}{n}\int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{\infty} F_{\mathrm{M}}(v) v^2 dv,$$

where θ corresponds to $\cos\theta = \frac{v_{\parallel,0}}{v_0}$ and θ_c to the boundary between trapped and passing particles. Utilising the calculations from b), the answer should be of the form $f_t = \sqrt{2r/R_0}$. What does this say about the balance between trapped and passing particles in a tokamak, i.e. where are the trapped particles most likely to dominate?

Solution 3.

(a) On the LFS the magnetic field is at its minimum, B_{min}, and the particle velocity is $v_0^2 = v_{\perp 0}^2 + v_{\parallel 0}^2$, and similarly on the HFS B_{max} and $v^2 = v_\perp^2 + v_\parallel^2$. Using the conservation of energy $E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 \leftrightarrow v_0^2 = v^2$. Further, considering the conservation of magnetic moment $\left(\mu = \frac{mv_\perp^2}{2B}\right)$:

$$\frac{mv_\perp^2}{2B_{max}} = \frac{mv_{\perp 0}^2}{2B_{min}} \rightarrow \frac{v_\perp^2}{B_{max}} = \frac{v_{\perp 0}^2}{B_{min}} \rightarrow \frac{v^2 - v_\parallel^2}{B_{max}} = \frac{v_0^2 - v_{\parallel 0}^2}{B_{min}} = \frac{v_0^2 - v_\parallel^2}{B_{max}}$$

$$\rightarrow v_\parallel^2 = v_0^2 - \frac{B_{max}}{B_{min}}\left(v_0^2 - v_{\parallel 0}^2\right)$$

The trapping condition states that the parallel velocity reaches zero at the HFS:

$$\rightarrow v_0^2 - \frac{B_{max}}{B_{min}}\left(v_0^2 - v_{\parallel 0}^2\right) < 0$$

$$\rightarrow \frac{v_{\parallel 0}}{v_0} < \sqrt{1 - \frac{B_{min}}{B_{max}}}$$

(b) The magnetic field has an 1/R dependence:

$$\rightarrow B_{min} = B_0\frac{R_0}{R_0 + r}, \quad B_{max} = B_0\frac{R_0}{R_0 - r} \leftrightarrow \frac{v_{\parallel 0}}{v_0} < \sqrt{1 - \frac{R_0 - r}{R_0 + r}} \approx \sqrt{2\frac{r}{R_0}}$$

since

$$1 - \frac{R_0 - r}{R_0 + r} = \frac{R_0 - r}{R_0 + r} - \frac{R_0 - r}{R_0 + r} = \frac{2r}{R_0 + r} \approx \frac{2r}{R_0 + r}, \text{ since } R_0 >> r.$$

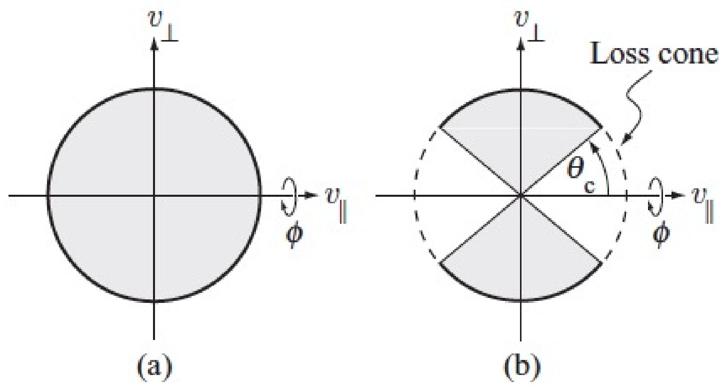


Figure 8.17 Velocity phase space showing: (a) a full, isotropic Maxwellian and (b) a Maxwellian with a loss cone.

Figure 5: Illustration of the loss cone in the velocity space [J.P. Freidberg, *Plasma Physics and Fusion Energy*, Cambridge University Press, 2007].

(c) The trapped fraction is obtained by integrating over the trapped portion of the distribution function. Assuming a Maxwellian velocity distribution normalized to integrate to 1, this gives

$$
\begin{aligned}
f &= \int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{\infty} F_{\mathrm{M}}(v) v^2 dv \\
&= \int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{\infty} \left(\frac{m}{2\pi T}\right)^{3/2} e^{-\frac{mv^2}{2T}} v^2 dv \\
&= \int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{\infty} \frac{1}{\pi^{3/2}} e^{-x^2} x^2 dx \\
&= \int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{\infty} \frac{2\pi}{\pi^{3/2}} e^{-x^2} x^2 dx \\
&= \int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{\infty} \frac{2}{\sqrt{\pi}} e^{-x^2} x^2 dx = \frac{1}{2} \int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \\
&= (-\cos(\pi-\theta_c) + \cos(\theta_c)) \frac{1}{2} = \cos\theta_c = \sqrt{2\frac{r}{R_0}},
\end{aligned}
$$

where integration by parts and the Gaussian integral has been used for the integration w.r.t. x. Assuming an example reactor of $R_0 = 3$ and $r = 1$ gives $f \sim 80$ %. Increasing the R_0 to 5 m leads to $f \sim 60$ %.

Exercise 4.

Maximum allowable current density in a tokamak

The helicity of the magnetic field in a tokamak is typically described with the safety factor, q, which is defined as the number of full toroidal loops that the magnetic field completes during a single poloidal revolution. The name stems from the fact that with certain q values, the plasma becomes susceptible to magnetohydrodynamic (MHD) instabilities, which may eventually lead to disruption and shut down of the reactor.

Find the maximum allowable current density in the tokamak, such that the minimum edge safety factor, in this case $q > 1$, is still within operational boundaries imposed by plasma instabilities. Assume the toroidal magnetic field to be $B_T \sim 7$ T, and the major radius to be $R \sim 7$ m. Assume a circular symmetric poloidal cross-section, large aspect-ratio tokamak, such that the safety factor can be approximated by the equation:

$$q = \frac{rB_T}{RB_\theta},$$

where r is the local minor radius of the tokamak, B_T is the total toroidal field magnitude, R is the major radius of the tokamak, and B_θ is the total magnitude of the poloidal field. The poloidal field is induced by a current running through the plasma, and so depends on the current density.

Solution 4.

The poloidal magnetic field can be calculated using Ampere's Law:

$$B_\theta = \frac{\mu_0 J_p \pi r^2}{2\pi r}.$$

Using the above expression and the safety factor approximation

$$q = \frac{2B_\phi}{R\mu_0 J_p} \leftrightarrow J_p = \frac{2B_\phi}{R\mu_0 q}.$$

Since we are interested in the maximum current density allowable for $q > 1$ this equation gives, $J_p < 1.59$ MA m^{-2}. If the plasma cross-section is assumed to be about $\pi 2^2$ m^2 $\sim$ 12.56 m^2, the maximum total current would be approximately 20 MA.

[1] J. P. Freidberg, Plasma Physics and Fusion Energy, Cambridge University Press, 2007, p.484-485