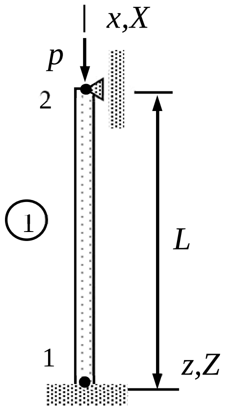
MEC-E8001 Finite Element Analysis, week 5/2023

1. Virtual work expression of a beam, which takes into account the bar and bending modes and the coupling between them, gives a non-linear equation system for the axial and transverse displacement. Determine the critical load p_{cr} causing the beam to buckle if the equation system is given by

$$-\frac{1}{5L^3}\begin{Bmatrix} 5L^3p+5L^2EAu_{X2} \\ 60EIu_{Z2}+6LEAu_{X2}u_{Z2} \end{Bmatrix}=0 .$$

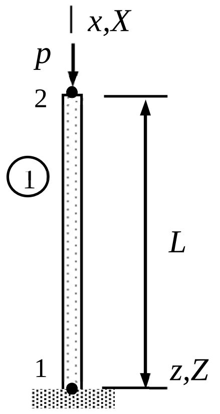
Answer $p_{\mathrm{cr}}=10\dfrac{EI}{L^2}$

2. Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the $xz-$plane. Parameters E, A, and I are constants.



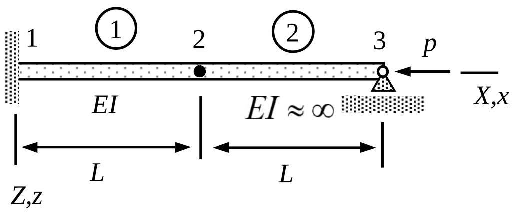
Answer $p_{\mathrm{cr}}=30\dfrac{EI}{L^2}$, $w=k(\dfrac{x}{L})^2(1-\dfrac{x}{L})$ (k is arbitrary)

3. Determine the buckling force p_{cr} of the beam shown by using one beam element. Displacements are confined to the $xz-$plane. The cross-section and material properties A, I, and E are constants.



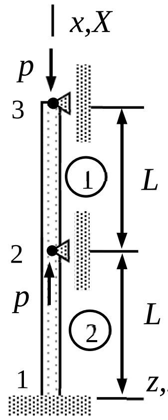
Answer $p_{\mathrm{cr}}=\dfrac{EI}{L^2}\dfrac{4}{3}\left(13-2\sqrt{31}\right)$

4. The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the $xz-$plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



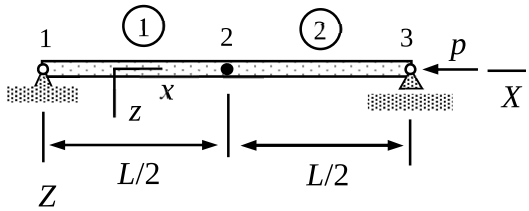
Answer $p_{\mathrm{cr}}=\dfrac{420}{23}\dfrac{EI}{L^2}$

5. Beam structure of the figure is loaded by opposite forces of magnitude p acting on nodes 2 and 3. Determine the buckling force p_{cr} of the structure using two beam elements. Displacements are confined to the xz – plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



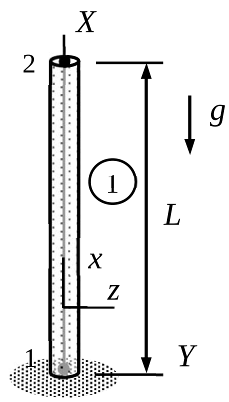
Answer $p_{\text{cr}} = 20\dfrac{EI}{L^2}$

6. The simply supported uniform beam shown is divided into two identical beam elements, each of length $L/2$. Displacements are confined to the xz – plane. Cross-section properties are A and I and Young's modulus of the material E. Determine the buckling load p_{cr}. Assume that rotation angles satisfy $\theta_{Y1} = -\theta_{Y3}$ and $\theta_{Y2} = 0$.



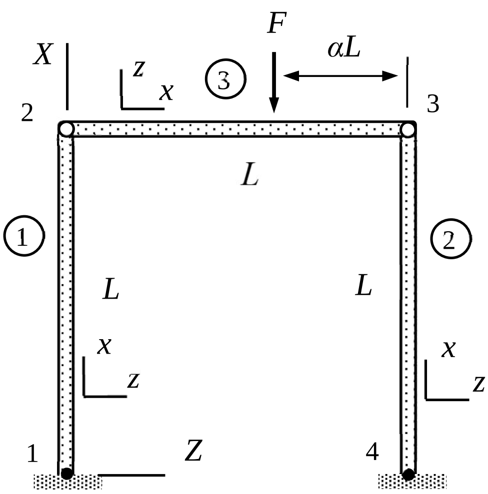
Answer $p_{\text{cr}} = \dfrac{240}{13+2\sqrt{31}}\dfrac{EI}{L^2}$

7. Find the density ρ_{cr} causing the beam of the figure to buckle in xz – plane. Start with the virtual work density taking into account the interaction of the bar and bending modes. Choose first $\delta w = 0$ in the virtual work density to solve for the axial displacement and the axial force N. After that, choose $\delta u = 0$ to find the virtual work expression taking into account the internal and coupling parts.



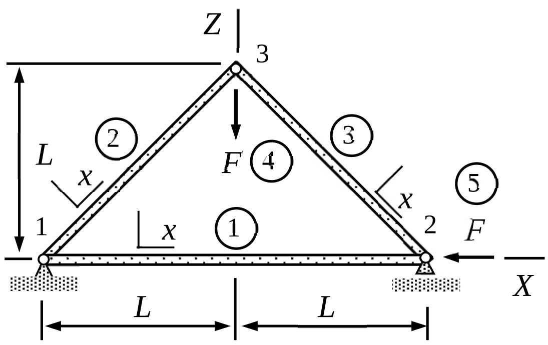
Answer $\rho_{\text{cr}} = \dfrac{120}{13+2\sqrt{31}}\dfrac{EI}{AgL^3}$

8. The plane frame of the figure consists of a rigid body 3 and beam elements 1 and 2. The joints at nodes 2 and 3 are frictionless. Determine the critical value of the force F acting on the rigid body at distance αL $\alpha \in [0,1]$ from node 3 making the frame to buckle laterally. The cross-section properties A and I and Young's modulus of the material E are constants.



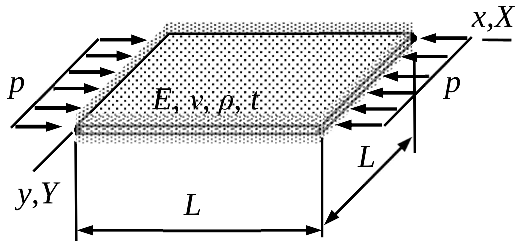
Answer $F = \dfrac{EI}{L^2}\dfrac{8}{3}(13-2\sqrt{31}) \approx 4.97\dfrac{EI}{L^2}$

9. Determine the critical value of force F causing some beam of the truss shown to buckle. First, use the bar model to solve for the nodal displacements and thereby the axial forces as functions of the loading F (assumed to be positive). After that, use criterion $N(F)=\pi^2 EI/h^2$ to find the first beam to buckle and the critical value F_{cr}. Cross-sectional areas of beams 2 and 3 are $\sqrt{8}A$ and that of beam 1 $2A$. The second moments of cross-sections I and the Young's modulus E of the material are constants.



Answer $F_{cr}=\frac{1}{2}\pi^2\frac{EI}{L^2}$ (beam 1 buckles)

10. Determine the critical value of the in-plane loading p_{cr} making the plate shown to buckle. Use $w(x,y)=a_0\sin(\pi x/L)\sin(\pi y/L)$ as the approximation and assume that $N_{xx}=-p$, $N_{yy}=0$, and $N_{xy}=0$. Problem parameters E, ν, ρ and t are constants. Integrals of sin and cos satisfy



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij} \text{ and } \int_0^L \cos(i\pi\frac{x}{L})\cos(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij}.$$

Answer $p_{cr}=4\frac{t^3}{12}\frac{E}{1-\nu^2}(\frac{\pi}{L})^2$

Virtual work expression of a beam, which takes into account the bar, bending, and the coupling of the modes, gives a non-linear equation system for the axial and transverse displacement. Determine the critical load p_{cr} causing the beam to buckle if the equation system is given by

$$-\frac{1}{5L^3}\begin{Bmatrix} 5L^3 p + 5L^2 EAu_{X2} \\ 60EIu_{Z2} + 6LEAu_{X2}u_{Z2} \end{Bmatrix} = 0 \,.$$

Solution

Although the equation system is non-linear, it can be solved in two steps for the critical load. Finding the normal forces (or axial displacements) as function of the load parameter is the first step. The first equation gives

$$5L^3 p + 5L^2 EAu_{X2} = 0 \quad \Leftrightarrow \quad \frac{EA}{L}u_{X2} + p = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{pL}{EA} \,.$$

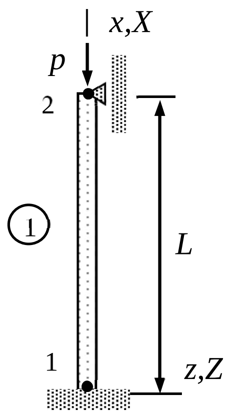
With this expression, the second equation simplifies to

$$60EIu_{Z2} + 6LEAu_{X2}u_{Z2} = 0 \;\Rightarrow\; (60EI - 6LEA\frac{pL}{EA})u_{Z2} = 0 \quad \Leftrightarrow \quad (60EI - 6L^2 p)u_{Z2} = 0$$

A non-trivial solution $u_{Z2} \neq 0$ is possible only if

$$60EI - 6L^2 p = 0 \quad \Leftrightarrow \quad p = 10\frac{EI}{L^2} \,. \leftarrow$$

Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the $xz-$plane. Parameters E, A, and I are constants.



Solution

The non-zero displacement/rotation components of the structure are $\theta_{y2} = \theta_{Y2}$ and $u_{x2} = u_{X2}$. In this case, the normal force in the beam $N = -p$ can be deduced without calculations on the axial displacement and it is enough to consider only the bending and coupling terms of the virtual work expression. As buckling is confined to the $xz-$plane

$$\delta W = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} - \frac{p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} \Leftrightarrow$$

$$\delta W = -\delta\theta_{Y2} \left(\frac{EI}{L^3} 4L^2 - \frac{p}{30L} 4L^2 \right) \theta_{Y2} .$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\left(\frac{EI}{L^3} 4L^2 - \frac{p}{30L} 4L^2 \right) \theta_{Y2} = 0 .$$

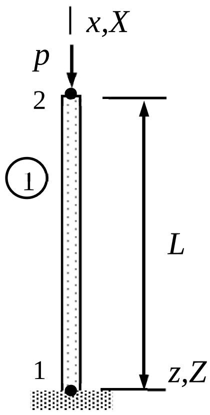
A non-trivial solution $\theta_{Y2} \neq 0$ is obtained only if

$$\frac{EI}{L^3} 4L^2 - \frac{p}{30L} 4L^2 = 0 \quad \Rightarrow \quad p_{cr} = 30 \frac{EI}{L^2} .$$

The shape function associated with θ_{Y2} is $N = -x^2 / L + x^3 / L^2$. Therefore, the buckled shape is given by (save an arbitrary multiplier)

$$w = -\frac{x^2}{L} + \frac{x^3}{L^2} .$$

Determine the buckling force p_{cr} of the beam shown by using one beam element. Displacements are confined to the xz–plane. The cross-section and material properties A, I, and E are constants.



Solution

The non-zero displacement/rotation components of the structure are $\theta_{y2}=\theta_{Y2}$, $u_{z2}=u_{Z2}$ and $u_{x2}=u_{X2}$. As the normal force in the beam

$$N=-p$$

can be deduced directly, it is enough to consider only the bending and coupling terms of the virtual work expression. For buckling in the xz–plane

$$\delta W=-\begin{Bmatrix}0\\0\\\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\-6L & 4L^2 & 6L & 2L^2\\-12 & 6L & 12 & 6L\\-6L & 2L^2 & 6L & 4L^2\end{bmatrix}-\frac{p}{30L}\begin{bmatrix}36 & -3L & -36 & -3L\\-3L & 4L^2 & 3L & -L^2\\-36 & 3L & 36 & 3L\\-3L & -L^2 & 3L & 4L^2\end{bmatrix}\right)\begin{Bmatrix}0\\0\\u_{Z2}\\\theta_{Y2}\end{Bmatrix}\Leftrightarrow$$

$$\delta W=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & 6L\\6L & 4L^2\end{bmatrix}-\frac{p}{30L}\begin{bmatrix}36 & 3L\\3L & 4L^2\end{bmatrix}\right)\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply (notice the scaling of the rotation and the force which make the two matrices dimensionless and simplify the eigenvalue calculations)

$$\left(\begin{bmatrix}12 & 6\\6 & 4\end{bmatrix}-\lambda\begin{bmatrix}36 & 3\\3 & 4\end{bmatrix}\right)\begin{Bmatrix}u_{Z2}\\L\theta_{Y2}\end{Bmatrix}=0 \text{ where } \lambda=\frac{pL^2}{30EI}.$$

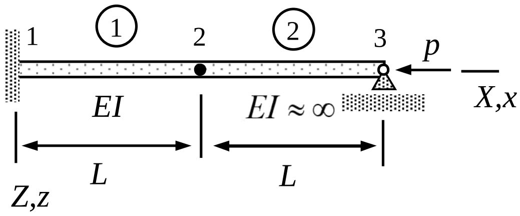
A non-trivial solution is obtained only if

$$\det\left(\begin{bmatrix}12 & 6\\6 & 4\end{bmatrix}-\lambda\begin{bmatrix}36 & 3\\3 & 4\end{bmatrix}\right)=(12-36\lambda)(4-4\lambda)-(6-3\lambda)^2=0 \quad\Rightarrow\quad \lambda=\frac{2}{45}\left(13\pm2\sqrt{31}\right).$$

The smallest of the values is the critical one

$$p_{\text{cr}}=\frac{EI}{L^2}\frac{60}{45}(13-2\sqrt{31})\approx2.48\frac{EI}{L^2}.$$

The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz-plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction between bar and bending modes

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

Also, as beam 2 is considered as rigid, virtual work expression of internal forces vanish and modelling uses a kinematical constraints $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$ and $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$. Let us choose A to be node 3 and B as node 2. Then

$$u_{Z2} = \theta_{Y3} L \quad \text{and} \quad \theta_{Y2} = \theta_{Y3}.$$

Although axial displacement is non-zero, it is not needed as the axial force in the structure $N = -p$ (negative means compression) can be deduced without calculations on the axial displacement.

The internal force and coupling parts of beam 1 take the forms ($u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2} = \theta_{Y3} L$, $\theta_{y2} = \theta_{Y2} = \theta_{Y3}$)

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = -\delta\theta_{Y3} 28 \frac{EI}{L} \theta_{Y3},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = \delta\theta_{Y3} \frac{46}{30} pL\theta_{Y3}.$$

Virtual work expression of the structure is the sum of the internal and stability parts

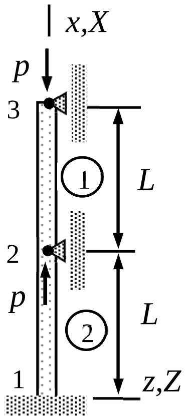
$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta\theta_{Y3}(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3} = 0.$$

In stability analysis the aim is to find a non-zero equilibrium solution so $\theta_{Y3} \neq 0$ and the multiplier needs to vanish for a solution. Therefore, the buckling load is given by

$$p_{\text{cr}} = \frac{30}{46}28\frac{EI}{L^2} = \frac{420}{23}\frac{EI}{L^2}.$$ ←



Beam structure of the figure is loaded by opposite forces of magnitude p acting on nodes 2 and 3. Determine the buckling force p_{cr} of the structure using two beam elements. Displacements are confined to the $xz-$ plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.

Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction modes. The non-zero displacement/rotation components of the structure are θ_{Y2} and θ_{Y3}.

For beam 1, $\theta_{y2} = \theta_{Y2}$ and $\theta_{y3} = \theta_{Y3}$. The axial force acting on the element $N = -p$ (negative means compression) follows from a free-body diagram. Therefore

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = \begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{pL}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}$$

giving

$$\delta W^1 = \delta W^{\mathrm{int}} + \delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right) \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

For beam 2, $\theta_{y2} = \theta_{Y2}$ and the axial force $N = 0$. Therefore $\delta W^{\mathrm{sta}} = 0$ and

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Virtual work expression of structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L} \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right) \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\left(\frac{EI}{L}\begin{bmatrix}8 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y3}\end{Bmatrix}=0\,.$$

In stability analysis the aim is to find the critical values (smallest of them typically) of the load parameter p such that the solution becomes non-unique. As the equilibrium equations are homogeneous, non-zero solution is obtained only if the matrix (above) is singular:

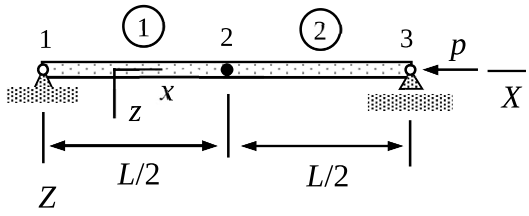
$$\det\left(\frac{EI}{L}\begin{bmatrix}8 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)=\left(8\frac{EI}{L}-4\frac{pL}{30}\right)\left(4\frac{EI}{L}-4\frac{pL}{30}\right)-\left(2\frac{EI}{L}+\frac{pL}{30}\right)^2=0 \quad \Rightarrow$$

$$\frac{pL^2}{EI}\in\{20,84\}\,.$$

The smallest of the values is the critical one

$$p_{\mathrm{cr}}=20\frac{EI}{L^2}\,.$$

The simply supported uniform beam shown is divided into two identical beam elements, each of length $L/2$. Displacements are confined to the xz – plane. Cross-section properties are A and I and Young's modulus of the material E. Determine the buckling load p_{cr}. Assume that rotation angles satisfy $\theta_{Y1} = -\theta_{Y3}$ and $\theta_{Y2} = 0$.



Solution

The axial force in the beams follows directly from a free body diagram and it is enough to consider the virtual work associated with bending and interaction modes. The non-zero displacements and rotations of the structure are θ_{Y1}, u_{Z2} and $\theta_{Y3} = -\theta_{Y1}$.

For beam 1, $\theta_{y1} = \theta_{Y1}$ and $u_{z2} = u_{Z2}$. The axial force $N = -p$ (negative means compression) follows from a free-body diagram. Here $h = L/2$

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y1} \\ u_{Z2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} 8\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y1} \\ u_{Z2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}$$

giving

$$\delta W^1 = \delta W^{\mathrm{int}} + \delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (8\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

For beam 2, $u_{z2} = u_{Z2}$, $\theta_{y3} = \theta_{Y3} = -\theta_{Y1}$ and the axial force $N = -p$. As $h = L/2$

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{Z2} \\ 0 \\ 0 \\ -\delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ 0 \\ 0 \\ -\theta_{Y1} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} 8\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{Z2} \\ 0 \\ 0 \\ -\delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ 0 \\ 0 \\ -\theta_{Y1} \end{Bmatrix} = \begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix},$$

so that

$$\delta W^2 = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\text{T}} (8\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

Virtual work expression of structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\text{T}} (16\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{2p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(16\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{2p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix} = 0.$$

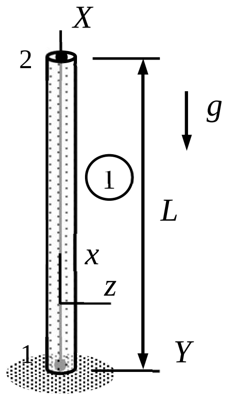
As the equilibrium equations are homogeneous, non-zero solution is obtained only if the matrix (above) is singular:

$$\det(16\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{2p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix}) = 0 \quad \Rightarrow \quad \frac{pL^2}{EI} = \frac{16}{3}\left(13 \pm 2\sqrt{31}\right).$$

The smallest of the values is the critical one

$$p_{\text{cr}} = \frac{16}{3}\left(13 - 2\sqrt{31}\right)\frac{EI}{L^2} = \frac{240}{13 + 2\sqrt{31}}\frac{EI}{L^2} \approx 9.94\frac{EI}{L^2}.$$

Find the density ρ_{cr} causing the beam of the figure to buckle in $xz-$plane. Start with the virtual work density taking into account the interaction of the bar and bending modes. Choose first $\delta w=0$ in the virtual work density to solve for the axial displacement and the axial force N. After that, choose $\delta u=0$ to find the virtual work expression taking into account the internal and coupling parts.



Solution

The displacements/rotations of the structure are $u_{x2}=u_{X2}$, $u_{z2}=u_{Y2}$ and $\theta_{y2}=-\theta_{Z2}$. The starting point is the virtual work density

$$\delta w_{\Omega}=-\frac{d\delta u}{dx}EA\frac{du}{dx}-\frac{d^2\delta w}{dx^2}EI_{zz}\frac{d^2 w}{dx^2}-N\frac{d\delta w}{dx}\frac{dw}{dx}+\delta u f_x \quad \text{where} \quad N=EA\frac{du}{dx}$$

which takes into account the bar and bending modes and their interaction. Approximations to axial displacement u and transverse displacement w ($\xi=x/h$ and $h=L$) are

$$u=\frac{x}{L}u_{X2} \quad \Rightarrow \quad \frac{du}{dx}=\frac{1}{L}u_{X2},$$

$$w=\begin{Bmatrix}(3-2\frac{x}{L})(\frac{x}{L})^2\\ L(\frac{x}{L})^2(\frac{x}{L}-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y2}\\ \theta_{Z2}\end{Bmatrix} \quad \Rightarrow \quad \frac{dw}{dx}=\begin{Bmatrix}\frac{6x}{L^2}-\frac{6x^2}{L^3}\\ \frac{3x^2}{L^2}-2\frac{x}{L}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y2}\\ \theta_{Z2}\end{Bmatrix} \quad \Rightarrow \quad \frac{d^2w}{dx^2}=\begin{Bmatrix}\frac{6}{L^2}-\frac{12x}{L^3}\\ \frac{6x}{L^2}-\frac{2}{L}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y2}\\ \theta_{Z2}\end{Bmatrix}.$$

In the first step $\delta w=0$. When the approximation to u is substituted there, virtual work density simplifies to

$$\delta w_{\Omega}=\delta w_{\Omega}=-\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x=-\frac{\delta u_{X2}}{L}EA\frac{u_{X2}}{L}-\delta u_{X2}\frac{x}{L}\rho g A \quad \Rightarrow$$

$$\delta W=\int_0^L \delta w_{\Omega}dx=-\delta u_{X2}(\frac{EA}{L}u_{X2}+\frac{L}{2}\rho g A).$$

Principle of virtual work and the fundamental lemma of variation calculus imply that (notice that the actual axial force is linear)

$$u_{X2}=-\frac{L^2}{2}\frac{\rho g}{E} \quad \text{giving as the axial force} \quad N=EA\frac{du}{dx}=\frac{EA}{L}u_{X2}=-\frac{L}{2}\rho g A.$$

In the second step $\delta u=0$. When the approximation to w is substituted there, the virtual work density becomes (virtual work expression is available also in the formulae collection)

$$\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{zz} \frac{d^2 w}{dx^2} - N \frac{d\delta w}{dx}\frac{dw}{dx} \quad \Rightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix} \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \dfrac{6}{L^2} - \dfrac{12x}{L^3} \\ \dfrac{6x}{L^2} - \dfrac{2}{L} \end{Bmatrix} EI \begin{Bmatrix} \dfrac{6}{L^2} - \dfrac{12x}{L^3} \\ \dfrac{6x}{L^2} - \dfrac{2}{L} \end{Bmatrix}^{\mathrm{T}} + \begin{Bmatrix} \dfrac{6x}{L^2} - \dfrac{6x^2}{L^3} \\ \dfrac{3x^2}{L^2} - 2\dfrac{x}{L} \end{Bmatrix} N \begin{Bmatrix} \dfrac{6x}{L^2} - \dfrac{6x^2}{L^3} \\ \dfrac{3x^2}{L^2} - 2\dfrac{x}{L} \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} \Rightarrow$$

$$\delta W = \int_0^L \delta w_\Omega dx = -\begin{Bmatrix} \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \frac{N}{30L} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply (with $N = -L\rho g A / 2$)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{\rho g A}{60} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} = 0.$$

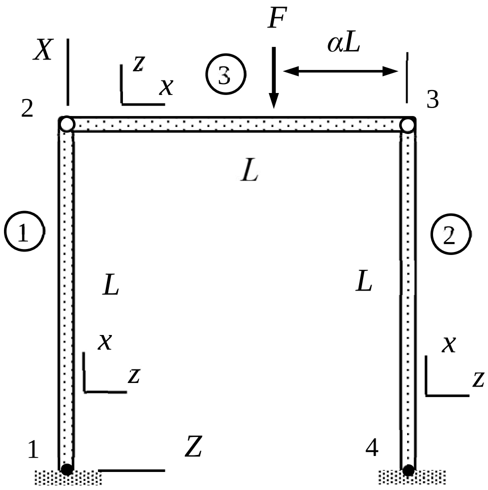
In stability analysis, the goal is to find the value of the loading parameter such that the solution is not unique. This is possibly only if

$$\det\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{\rho g A}{60} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) = 0 \quad \Rightarrow \quad 12\left(\frac{EI}{L^2 Ag}\right)^2 - \frac{13}{5}\left(\frac{EI}{L^2 Ag}\right)(L\rho) + \frac{3}{80}(L\rho)^2 = 0$$

giving (the smallest ρ matters)

$$\rho = \frac{8}{3}(13 \pm 2\sqrt{31}) \frac{EI}{AgL^3} \quad \Rightarrow \quad \rho_{\mathrm{cr}} = \frac{8}{3}(13 - 2\sqrt{31}) \frac{EI}{AgL^3} = \frac{120}{13 + 2\sqrt{31}} \frac{EI}{AgL^3}. \leftarrow$$

The plane frame of the figure consists of a rigid body 3 and beam elements 1 and 2. The joints at nodes 2 and 3 are frictionless. Determine the critical value of the force F acting on the rigid body at distance αL $\alpha \in [0,1]$ from node 3 making the frame to buckle laterally. The cross-section properties A, I and Young's modulus of the material E are constants.



Solution

The non-zero displacement and rotation components are $u_{Z2} = u_{Z3}$, $\theta_{Y2} = \theta_{Y3}$, u_{X2}, and u_{X3}. The vertical contact forces between the beams at nodes 2 and 3 follow from the equilibrium equations of the rigid body 3. Therefore, the axial force in beam 1 is $N = -\alpha F$ and that in beam 2 $N = -(1-\alpha)F$ (both compression) and it is enough to consider only the bending of beams 1 and 2.

For beam 1, the non-zero displacement/rotation components are (omitting the axial one as only the bending mode is considered) $u_{z2} = u_{Z2}$ and $\theta_{y2} = \theta_{Y2}$

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha F}{30L} \begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}$$

therefore

$$\delta W^1 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{\alpha F}{30L} \begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\right) \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

For beam 2, the non-zero displacement/rotation components are (again omitting the axial one as only the bending mode is considered) $u_{z3} = u_{Z2} = u_{Z3}$ and $\theta_{y3} = \theta_{Y3} = \theta_{Y2}$

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{(1-\alpha)F}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}$$

therefore

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{(1-\alpha)F}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Virtual work expression of the structure is the sum of element contributions i.e

$$\delta W = \delta W^1 + \delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(2\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{F}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

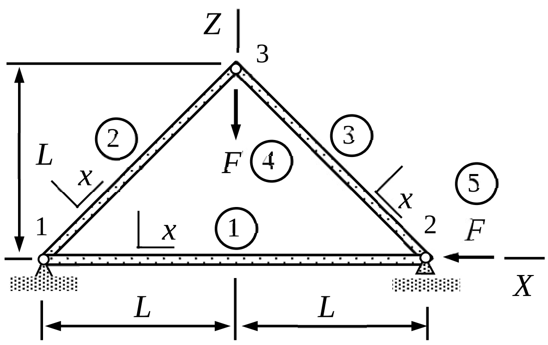
A non-trivial solution is possible only if the matrix inside parenthesis is singular

$$\det\left(\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{FL^2}{60EI}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\right) = 0 \quad \Rightarrow \quad \frac{FL^2}{EI} = \frac{8}{3}(13 \pm 2\sqrt{31}).$$

Critical value is the smallest of the solutions

$$F = \frac{EI}{L^2}\frac{8}{3}(13 - 2\sqrt{31}) \approx 4.97\frac{EI}{L^2}.$$

Determine the critical value of force F causing some beam of the truss shown to buckle. First, use the bar model to solve for the nodal displacements and thereby the axial forces as functions of the loading F (assumed to be positive). After that, use criterion $N(F)=\pi^2 EI/h^2$ to find the first beam to buckle and the critical value F_{cr}. Cross-sectional areas of beams 2 and 3 are $\sqrt{8}A$ and that of beam 1 $2A$. The second moments of cross-sections I and the Young's modulus E of the material are constants.



Solution

In the first step, the structure is considered as bar structure to find the nodal displacements as functions of the loading. Virtual work expression of the bar element needed in the displacement analysis

$$\delta W^{\mathrm{int}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \text{ and } \delta W^{\mathrm{ext}}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

depend on the cross-sectional area A, Young's modulus E, bar length h, force per unit length of the bar f_x in the direction of the $x-$axis. The non-zero displacement/rotation components of the structure are u_{X2}, u_{X3}, and u_{Z3}. Virtual work expression of the elements are (no distributed forces)

Bar 1: $u_{x1}=0$, $u_{x2}=u_{X2}$,

$$\delta W^1=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{E2A}{2L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix},$$

Bar 2: $u_{x1}=0$, $u_{x3}=\frac{1}{\sqrt{2}}(u_{X3}+u_{Z3})$

$$\delta W^2=-\begin{Bmatrix}0\\ \delta u_{X3}+\delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X3}+u_{Z3}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix}$$

Bar 3: $u_{x3}=\frac{1}{\sqrt{2}}(u_{X3}-u_{Z3})$, $u_{x2}=\frac{1}{\sqrt{2}}u_{X2}$

$$\delta W^3=-\begin{Bmatrix}\delta u_{X3}-\delta u_{Z3}\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X3}-u_{Z3}\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix}$$

Force 4: $\delta W^4=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ 0\\ F\end{Bmatrix}$

Force 5: $\delta W^5 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}$

Virtual work expression of a structure is sum of the element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \\ F \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply the linear equation system and thereby the solution to nodal displacements

$$\frac{EA}{L} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} + F \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} = \frac{FL}{EA} \begin{Bmatrix} -1/2 \\ -1/4 \\ -1/4 \end{Bmatrix}.$$

The axial forces of the beams become (notice that the expression depends on the displacement components in the material coordinate systems of the beams)

Beam 1: $N = \frac{EA}{L} u_{X2} = \frac{EA}{L}\left(-\frac{1}{2}\frac{FL}{EA}\right) = -\frac{1}{2}F$,

Beam 2: $N = \frac{E\sqrt{8}A}{\sqrt{2}L}\frac{1}{\sqrt{2}}(u_{X3} + u_{Z3}) = -\frac{1}{\sqrt{2}}F$,

Beam 3: $N = \frac{E\sqrt{8}A}{\sqrt{2}L}\frac{1}{\sqrt{2}}(u_{X2} - u_{X3} + u_{Z3}) = -\frac{1}{\sqrt{2}}F$.

The critical loading of the truss as predicted by criterion $N = \pi^2 EI / h^2$ in which N is the magnitude of the compressive axial force

Beam 1: $F = \frac{1}{2}\pi^2 \frac{EI}{L^2} \approx 4.93 \frac{EI}{L^2}$,

Beam 2: $F = \frac{1}{\sqrt{2}}\pi^2 \frac{EI}{L^2} \approx 6.98 \frac{EI}{L^2}$,

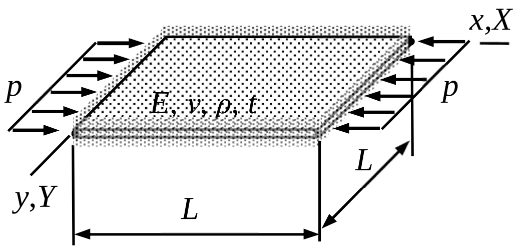
Beam 3: $F = \frac{1}{\sqrt{2}}\pi^2 \frac{EI}{L^2} \approx 6.98 \frac{EI}{L^2}$.

The critical load of the truss is the smallest of the critical loads calculated for the beams

$$F_{\mathrm{cr}} = \frac{1}{2}\pi^2 \frac{EI}{L^2} \approx 4.93 \frac{EI}{L^2}. \quad \leftarrow$$

Beam 1 is likely to buckle first.

Determine the critical value of the in-plane loading p_{cr} making the plate shown to buckle. Use $w(x,y)=a_0\sin(\pi x/L)\sin(\pi y/L)$ as the approximation and assume that $N_{xx}=-p$, $N_{yy}=0$, and $N_{xy}=0$. Problem parameters E, ν, ρ and t are constants. Integrals of sin and cos satisfy, e.g.,



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij} \text{ and } \int_0^L \cos(i\pi\frac{x}{L})\cos(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij}.$$

Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_\Omega^{\mathrm{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\quad \delta w_\Omega^{\mathrm{sta}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives are

$$w(x,y)=a_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad\Rightarrow$$

$$\frac{\partial w}{\partial x}=a_0(\frac{\pi}{L})\cos(\pi\frac{x}{L})\sin(\pi\frac{y}{L}),\quad \frac{\partial w}{\partial y}=a_0(\frac{\pi}{L})\sin(\pi\frac{x}{L})\cos(\pi\frac{y}{L}),$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial y^2}=-a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}),\quad \frac{\partial^2 w}{\partial x\partial y}=a_0(\frac{\pi}{L})^2\cos(\pi\frac{x}{L})\cos(\pi\frac{y}{L}).$$

When the approximation is substituted there, virtual work densities of the internal and forces and that of the coupling simplify to ($N_{xx}=-p$ and $N_{yy}=N_{xy}=0$)

$$\delta w_\Omega^{\mathrm{int}}=-\delta a_0\frac{t^3E}{12(1-\nu^2)}(\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L})\sin^2(\frac{\pi y}{L})(1+\nu)+(1-\nu)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})]a_0,$$

$$\delta w_\Omega^{\mathrm{sta}}=\delta a_0 p(\frac{\pi}{L})^2\cos^2(\pi\frac{x}{L})\sin^2(\pi\frac{y}{L})a_0.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate/element

$$\delta W^{\rm int} = \int_0^L \int_0^L \delta w_\Omega^{\rm int} dxdy = -\delta a_0 \frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 \, ,$$

$$\delta W^{\rm sta} = \int_0^L \int_0^L \delta w_\Omega^{\rm sta} dxdy = \delta a_0 \, p(\frac{\pi}{L})^2 (\frac{L}{2})^2 a_0 \, .$$

Virtual work expression

$$\delta W = \delta W^{\rm int} + \delta W^{\rm sta} = -\delta a_0 [\frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 - p(\frac{\pi}{L})^2 (\frac{L}{2})^2] a_0 \, .$$

Principle of virtual work $\delta W = 0 \;\forall \delta a_0$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 [\frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 - p(\frac{\pi}{L})^2 (\frac{L}{2})^2] a_0 = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

$$[\frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 - p(\frac{\pi}{L})^2 (\frac{L}{2})^2] a_0 = 0 \, .$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\rm cr} = \frac{1}{3} \frac{t^3 E}{1-\nu^2} (\frac{\pi}{L})^2 \, . \quad \leftarrow$$