PHYS-E0463 Fusion Energy Technology **DL Feb 8th, 2023**
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General information

The exercise sessions will be held as blackboard sessions, where the participants will present their solutions to the group. As such, the problems should be set up and solved before the session. The focus of the exercises lies on analyzing and discussing the task at hand together with the group: thus, a perfect solution is not required to be awarded points. A point will be awarded for each question, and a person will be chosen to present their solution from the pool.

Exercise 1.

Equilibrium of magnetic fusion devices

The equations describing the equilibrium properties of all magnetic configurations of fusion interest are [1]:

MHD momentum equation (Newton II): $$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{J}\times\mathbf{B} - \nabla p \rightarrow \mathbf{J}\times\mathbf{B} = \nabla p \tag{1}$$

Ampère's law: $$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{2}$$

Gauss' Law for magnetism: $$\nabla\cdot\mathbf{B} = 0, \tag{3}$$

where ρ is the density, $\mathbf{v}$ the velocity, $\mathbf{J}$ the current density, $\mathbf{B}$ the magnetic field, and p the plasma pressure. The equivalence in the first equation is due to the assumption that the plasma is stationary ($\mathbf{v} = 0$). Equilibria with flows do exist, but they are not considered here.

(a) Using equation (1), show that the magnetic field lines must lie in surfaces of constant pressure, i.e. that the pressure gradient is perpendicular to the magnetic field $\mathbf{B}$.

(b) Using equation (1) and a similar method as in a), show that the current lines must lie in surfaces of constant pressure.

(c) Using equations (1)-(3), derive equation

$$\nabla_\perp\left(p + \frac{B^2}{2\mu_0}\right) - \boldsymbol{\kappa}\frac{B^2}{\mu_0} = 0. \tag{4}$$

Use Ampère's law to eliminate $\mathbf{J}$ from the MHD momentum balance equation. You will also need the following:

$$\nabla p = \nabla_\perp p + \nabla_\parallel p \approx \nabla_\perp p \tag{5}$$

$$\nabla B^2 = \nabla_\parallel B^2 + \nabla_\perp B^2 \tag{6}$$

$$\nabla_\parallel B^2 = \mathbf{b}(\mathbf{b}\cdot\nabla)B^2 \tag{7}$$

$$(\nabla\times\mathbf{B})\times\mathbf{B} = -\frac{1}{2}\nabla B^2 + \frac{1}{2}\mathbf{b}(\mathbf{b}\cdot\nabla)B^2 + B^2(\mathbf{b}\cdot\nabla)\mathbf{b} \tag{8}$$

$\boldsymbol{\kappa} = (\mathbf{b} \cdot \nabla)\mathbf{b}$ is the curvature vector of the magnetic field, and $\mathbf{b} = \mathbf{B}/B$ is the unit vector of the magnetic field. $\boldsymbol{\kappa}$ points towards the local center of curvature of $\mathbf{B}$, and its magnitude is equal to the inverse radius of curvature.

(d) Shortly explain the physical interpretation of equation (4). The plasma pressure p represents the thermal energy of the plasma, but what are the meaning of the other two terms in the equation?

Solution 1.

(a) Using the equilibrium momentum equation we can show that the gradient of the thermal pressure of the plasma is aligned perpendicular to the magnetic field. Therefore, the magnetic flux surfaces represent surfaces of constant pressure.

$$\nabla p \cdot \mathbf{B} = (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{B} = 0.$$

(b) Using the equilibrium momentum equation it can be shown that the gradient of the thermal pressure of the plasma is perpendicular to the current lines in the plasma. Therefore, the current lines lie in surfaces of constant pressure.

$$\nabla p \cdot \mathbf{J} = (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{J} = 0.$$

(c) Using Ampère's law, the equation for the current density in the plasma can be written:

$$\mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0}.$$

Substituting this into equation (1) gives:

$$\mathbf{J} \times \mathbf{B} = \nabla p$$
$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$$

Using the vector identity $(\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{2}\nabla B^2 + \frac{1}{2}\mathbf{b}(\mathbf{b} \cdot \nabla)B^2 + B^2(\mathbf{b} \cdot \nabla)\mathbf{b}$ gives

$$\mu_0 \nabla p = -\frac{1}{2}\nabla B^2 + \frac{1}{2}\mathbf{b}(\mathbf{b} \cdot \nabla)B^2 + B^2(\mathbf{b} \cdot \nabla)\mathbf{b}$$

Now use the fact that $\nabla p = \nabla_\perp p$ and $\nabla_\parallel B^2 = \mathbf{b}(\mathbf{b} \cdot \nabla)B^2$

$$\mu_0 \nabla_\perp p = -\frac{1}{2}\nabla B^2 + \frac{1}{2}\nabla_\parallel B^2 + B^2(\mathbf{b} \cdot \nabla)\mathbf{b}$$
$$\mu_0 \nabla_\perp p = -\left(\frac{1}{2}\nabla B^2 - \frac{1}{2}\nabla_\parallel B^2\right) + B^2(\mathbf{b} \cdot \nabla)\mathbf{b}$$
$$\mu_0 \nabla p = -\nabla_\perp \left(\frac{B^2}{2}\right) + B^2\boldsymbol{\kappa}$$

Rearranging gives

$$\nabla_\perp \left(p + \frac{B^2}{2\mu_0}\right) - \boldsymbol{\kappa}\frac{B^2}{\mu_0} = 0.$$

(d) Equation (4) describes the pressure balance perpendicular to the magnetic field. The quantity p represents the plasma pressure, $B^2/2\mu_0$ the magnetic pressure, and the last term, $\boldsymbol{\kappa} B^2/\mu_0$, the tension force created by the curvature of the magnetic field lines. The plasma pressure can be thus balanced, depending on the exact configuration, by either a magnetic pressure gradient, magnetic tension, or both.

Exercise 2.
Pressure driven instabilities

Instabilities driven primarily by the pressure gradient in the plasma are typically classified either as *interchange modes* or *ballooning modes* [1]. Start by investigating the somewhat amusingly named *sausage mode.* Consider a straight plasma cylinder at equilibrium with a radius, a, a plasma pressure, p, and a current running only in the direction parallel to the axis of the cylinder. The axial magnetic field is assumed to be zero in this exercise (i.e. the magnetic field in the direction of the cylinder axis $B_z = 0$).

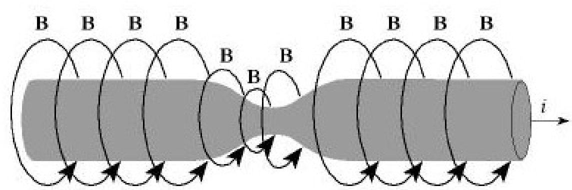

Figure 1: Sausage instability in a cylindrical plasma.

(a) Using Ampere's law, show that the poloidal magnetic field, B_θ, scales as $\propto 1/r$.

(b) Next, consider a radially narrowing, "sausage like", perturbation, as presented on figure Fig. 1. If we assume the pressure to remain constant, what happens to the equilibrium in the "waist" of the perturbation? How could you compensate for this instability?

Solution 2.

(a) The poloidal magnetic field outside the surface of the cylinder is given by the Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

which when integrated over the area of the cross-section of the cylinder gives

$$2\pi r B_\theta = \mu_0 I,$$

where I is the total current flowing through the cylinder and Stoke's theorem has been applied. Therefore, the magnitude of the poloidal magnetic field is given by

$$B_\theta = \frac{\mu_0 I}{2\pi r} \propto \frac{1}{r}.$$

(b) In a sausage like perturbation, the poloidal magnetic field at the surface of the cylinder increases in the waist and reduces in the bulge (as $\sim \frac{1}{r}$). For example in the figure (2), $B_3 > B_2 > B_1$. The magnetic tension, proportional to the square of magnetic field and inversely proportional to the radius of curvature, is thus reduced at the bulge, while it is increased at the waist. In a Z-pinch configuration, magnetic tension is the only confining force at the outer edge of the plasma column (see Freidberg, section 11.6.2). This means that the bulge tends to grow bigger and bigger, while the waist continues to shrink. This instability could be counteracted by introducing an axial magnetic field.

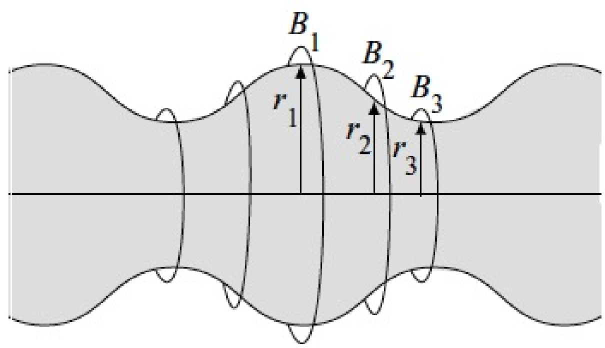

Figure 12.3 Schematic diagram of a Z-pinch interchange instability.

Figure 2: Illustration of the interchange mode[1].

Exercise 3.

Current-driven instabilities

Instabilities driven by the current density parallel to the magnetic field are often called *kink modes* [1], which will be treated next. Consider a pressure-less plasma cylinder ($p = 0$), where a surface current layer on the surface of the cylinder, $r = a$, separates a purely axial magnetic field inside the cylinder, $(0, 0, B_z)$, $r < a$, from a purely poloidal magnetic field outside the cylinder, $(0, B_\theta a/r, 0)$, $r > a$, (lecture slides 9 - 11 'Tokamaks Part B'). B_z and B_θ are constants.

(a) Consider a "kink like" perturbation of the plasma column, as in figure Fig. 3. What happens on the inside/outside of the bend? What are the stabilizing and destabilizing forces in this case?

(b) Consider a tokamak, which has both a poloidal and a toroidal field. Based on your reasoning in part a, which one of them is the stabilizing one against kink perturbations?

(c) The experimentally measured energy confinement time τ_E is nearly linearly proportional to the toroidal current in tokamaks. Why is it not a feasible strategy to simply increase the toroidal current to achieve a sufficient τ_E for ignition?

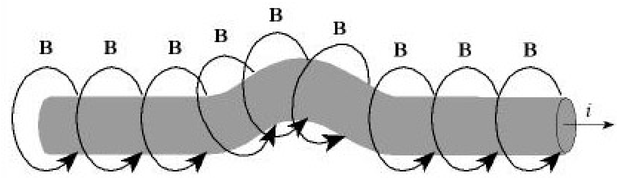

Figure 3: Kink mode in a cylindrical plasma.

Solution 3.

(a) In a kink-like perturbation, the poloidal magnetic field lines are squeezed at the inside of the bend, increasing the magnetic tension, and spread on the outside of the bend, reducing the magnetic tension. This tends to further enhance the perturbation. On the other, the axial field lines are also bent during the perturbation. The magnetic tension of the axial field provides a stabilizing mechanism.

(b) The axial (toroidal in tokamaks) magnetic field stabilizes the growth of the kink instability, since magnetic tension associated with curvature of the toroidal field increases during the perturbation.

(c) Since the maximum achievable toroidal magnetic field is determined by magnet engineering considerations, e.g. the need to maintain superconductivity and to avoid excess mechanical loads, the maximum toroidal current, I_p, is limited by the susceptibility of the plasma to kink modes at high toroidal currents.

Exercise 4.
Ideal modes, resistive modes, and edge localized modes.

(a) Can you explain the physical difference between ideal and resistive modes? How do the resistive modes impact the magnetic geometry, and what implications do they have for the operation of a tokamak?

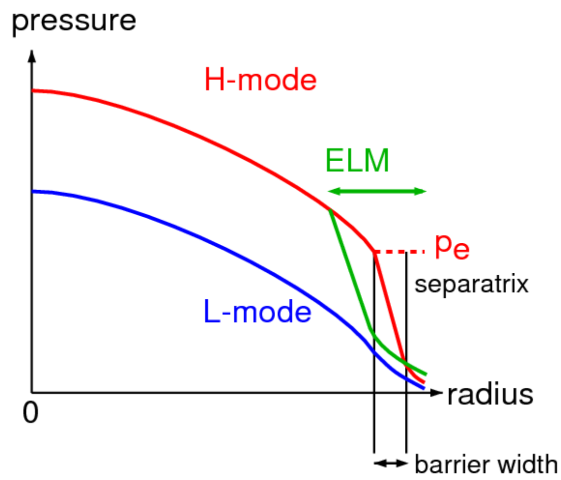

Figure 4: Plasma pressure profiles for L-mode (blue) and H-mode (red), and the associated ELM relaxation (green). Figure credit goes to DIFFER.

(b) Under certain conditions, the performance of a tokamak can improve suddenly when the heating power is increased. This transition from low confinement mode (L-mode) to high confinement mode (H-mode) is dubbed the L-H transition, and it is currently a crucial principle in achieving high pressures in tokamak plasmas.

As illustrated in figure 4, the transition is identified by a significant increase in plasma pressure in the core plasma and strong gradient in plasma pressure on the edge. This "edge transport barrier", however, is not inherently stable. Edge localized mode (ELM) instabilities lead to a (partial) relaxation of the gradient and thus a significant amount of heat is launched at the wall. After the energy is released, the gradient (the barrier) builds up again and this leads to a new relaxation event, creating a cyclical instability.

Type-I ELMs are the standard, large ELMs associated with a standard, high performance, high confinement mode plasmas. In an ITER-sized device with the expected edge plasma parameters in ITER, the extrapolated size of type-I ELMs can be about 20% of the energy stored in the steep gradient region at the edge, called the pedestal [2]. The melting temperature threshold of the divertor components in ITER determines the tolerable heat impact to be about $\Delta W/\sqrt{t} < 20 - 40$ MW $\mathrm{s}^{1/2}$ m^{-2}, where ΔW represents the power flux density impacting the surface component, and t is the characteristic time scale of the situation [3]. Estimate this heat impact parameter for a type-I ELM in ITER, assuming:

- the stored pedestal energy in ITER is about 120 MJ.
- 65 % of the ELM energy reaches the high heat flux areas in the divertor chamber.
- the effective plasma wetted area is about 3 m^3.
- the effective ELM duration is about 0.5 ms. A very nice review of ELMs in tokamaks can be found in [2].

What is your conclusion about type-I H-mode operation in ITER, can the device absorb the exhausted power?

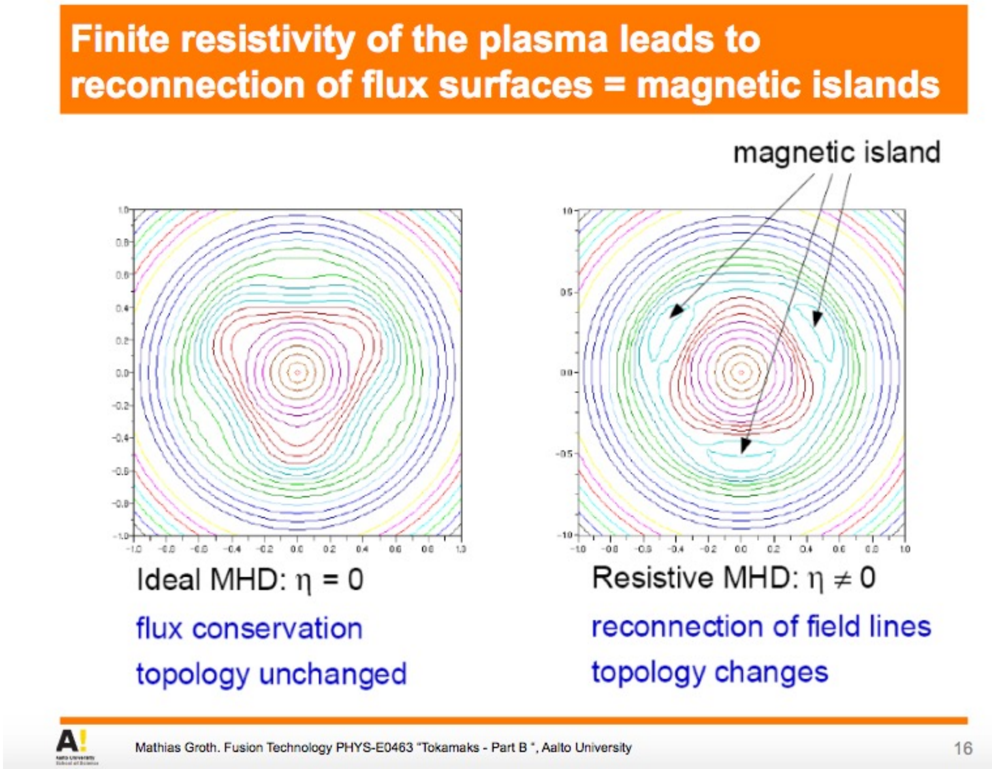

Figure 5: Illustration of the impact of the plasma resistivity on the magnetic topology.

Solution 4.

(a) In ideal MHD, the zero resistivity leads to a condition of the magnetic field lines being "frozen" in the plasma. This can be easily proven by considering a magnetic flux flowing through a surface of a plasma

$$\psi(t) = \int \mathbf{B} \cdot \mathbf{n} dS,$$

where S is an open surface area and $\mathbf{n}$ is the surface normal. Considering the time derivative of this flux leads to the equation

$$\frac{d\psi}{dt} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS + \oint \mathbf{B} \times \mathbf{u}_\perp \cdot d\mathbf{l} = -\int \nabla \times \mathbf{E} \cdot \mathbf{n} dS + \oint \mathbf{B} \times \mathbf{u}_\perp \cdot d\mathbf{l}.$$

where $\mathbf{u}_\perp$ is an arbitrary velocity of the boundary of the surface area, and Faraday's law has been applied for the second equivalence, eliminating the $\partial B/\partial t$-term. Applying Stokes' theorem, this can be written as the line-integral

$$\frac{d\psi}{dt} = -\oint [\mathbf{E} - \mathbf{B} \times \mathbf{u}_\perp] \cdot d\mathbf{l} = \oint [(\mathbf{v}_\perp - \mathbf{u}_\perp) \times \mathbf{B}] \cdot d\mathbf{l},$$

Where zero resistivity,

$$\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B} = 0,$$

has been assumed for the last equivalence (so called ideal Ohm's law). Therefore, the flux passing through any arbitrary cross section is conserved when the cross sectional area of the flux tube moves with the plasma ($\mathbf{u}_\perp = \mathbf{v}_\perp$). In other words, the magnetic field lines are "frozen" in the plasma. Therefore, the topology of the magnetic flux surfaces does not change during an ideal MHD instability (fig. 5).

In the case of finite resistivity, η, Ohm's law reads

$$\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B} = \eta \mathbf{j}.$$

One of the impacts of finite resistivity of the plasma is that the magnetic field lines can diffuse through the plasma. This can lead to reconnection of the magnetic field lines and formation of magnetic islands, and in so-called *tearing modes.* Since the plasma is not confined along the magnetic field lines, the islands enhance radial transport of plasma inside the device. This deteriorates the confinement properties of the plasmas. The tearing modes can lead to a disruption, if the island continues to grow in an uncontrolled fashion.

(b) The total elm energy loss is about $W_{\mathrm{ELM}} = W_{\mathrm{ped}} * 0.2 = 24$ MJ.

The amount of energy reaching the divertor chamber is about $W_{\mathrm{divertor}} = W_{\mathrm{ELM}} * 0.65 = 15.6$ MJ.

The effective energy flux in the divertor chamber was about $W_{\mathrm{divertor}}/A_{\mathrm{divertor}} = 5.2$ MJ.

The effective ELM impact on the divertor target is about $\frac{W_{\mathrm{divertor}}}{A_{\mathrm{divertor}\sqrt{\tau_{\mathrm{E}}}}} = 234\ \mathrm{MJm^{-2}s^{-1/2}}$.

This is way above the maximum limit of 20 $\mathrm{MJm^{-2}s^{-1/2}}$. Therefore, unmitigated type-I ELM operation is likely to be unacceptable in ITER.

References: [1] J.P. Freidberg, *Plasma Physics and Fusion Energy*, Cambridge University Press, 2007

[2] K. Kamiya, *et al.*, *Edge localized modes: recent experimental findings and related issues*, Plasma Phys. Control. Fusion, **49**, (2007), S43 – S62

[3] A. Herrmann, *Overview on stationary and transient divertor heat loads*, Plasma Phys. Control. Fusion, **44**, (2002), 883 – 903.