Aalto University
School of Electrical
Engineering

ELEC-C8201: Control and Automation

5. The Root Locus Method
&
PID Controllers

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In the previous lecture…

You:

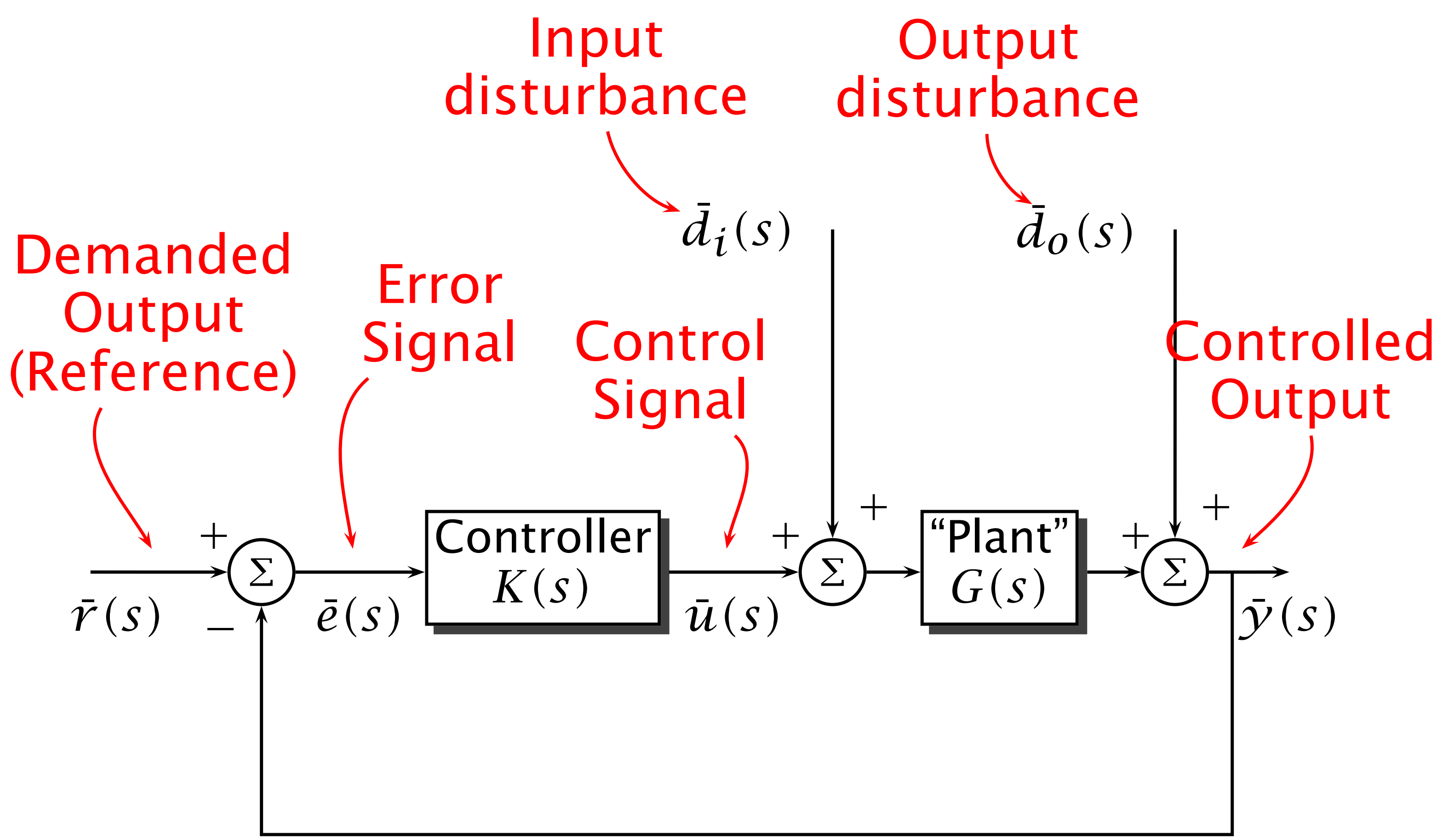
- Understood the concept of stability of dynamic systems
- Got introduced to the Routh-Hurwitz method as a tool for assessing system stability.

Learning outcomes

By the end of *this* lecture, you should be able to:

- Understand the concept of the root locus and its role in control system design
- Know how to obtain a root locus plot by sketching or using MATLAB
- Be familiar with the PID controller as a key element of many feedback systems

The closed-loop characteristic equation



- The dynamic performance of a closed-loop control system is described by the closed-loop transfer function

$$\bar{y}(s) = \frac{1}{1 + K(s)G(s)}\bar{d}_o(s) + \frac{G(s)}{1 + K(s)G(s)}\bar{d}_i(s) + \frac{K(s)G(s)}{1 + K(s)G(s)}\bar{r}(s)$$

- **Note:** All the closed-loop transfer functions have the same denominator.

- The closed loop poles are the roots of the closed-loop characteristic equation

$$1 + K(s)G(s) = 0$$

- The performance of a feedback system
  - Closed-loop stability of the system
  - Characteristics of the closed-loop system's transient response (e.g., speed of response, presence of any resonances etc)

  can be described in terms of the location of the roots of the characteristic equation in the s-plane.

- The poles lie in the s-plane, and given s is a complex is a complex variable, the characteristic equation may be rewritten in polar form as

$$|K(s)G(s)|\angle\big(K(s)G(s)\big) = -1 + j0$$

and therefore it is necessary that

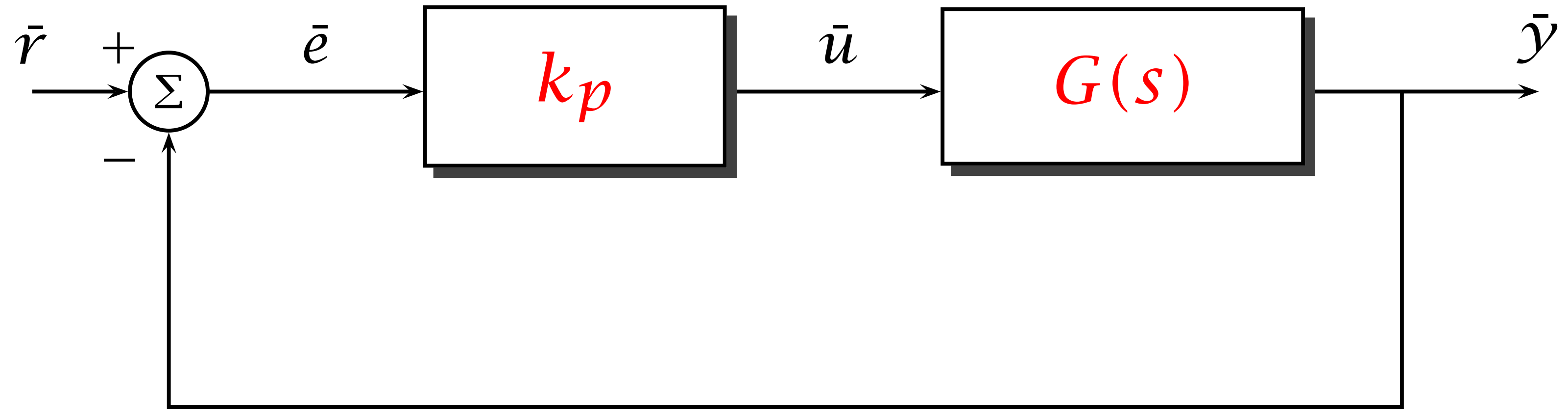
$$|K(s)G(s)| = 1 \qquad \text{and} \qquad \angle\big(K(s)G(s)\big) = 180^0 + k360^o$$

The root locus plot

- **Root locus plot:** a graph showing how the roots of the characteristic equation move around the s-plane as a *single* parameter varies
    - a powerful tool for designing and analyzing feedback control systems
    - frequently necessary to adjust one or more system parameters in order to obtain suitable root locations → determine how the roots of the characteristic equation of a given system migrate about the s-plane as the parameters are varied
    - provides the engineer with a measure of the sensitivity of the roots of the system to a variation in the parameter being considered
- The root locus plot may be used in conjunction with the Routh-Hurwitz criterion

The root locus concept - Proportional control

- In this case, $K(s) = k_p$



- Typical result of increasing the gain k_p, (for systems where $G(s)$ is stable):
  - ✓ Increased accuracy of control
  - – Increased control action
  - – Reduced damping (→ more oscillations)
  - – Possible loss of closed-loop stability for large gain k_p

Example 1:

- Consider the following critically damped 2nd-order system

$$G(s) = \frac{1}{(s+1)^2}$$

Then

$$\bar{y}(s) = \frac{k_p G(s)}{1 + k_p G(s)} \bar{r}(s) = \frac{k_p \frac{1}{(s+1)^2}}{1 + k_p \frac{1}{(s+1)^2}} \bar{r}(s)$$

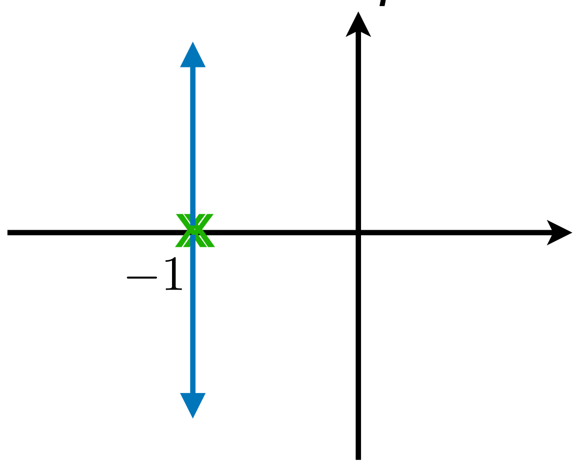
$$= \frac{k_p}{s^2 + 2s + 1 + k_p} \bar{r}(s)$$

The characteristic equation representing this system is

$$\Delta(s) = s^2 + 2s + 1 + k_p = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The closed-loop poles are $s = -1 \pm j\sqrt{k_p}$

"root locus plot"



Movement of closed-loop poles as k_p increases

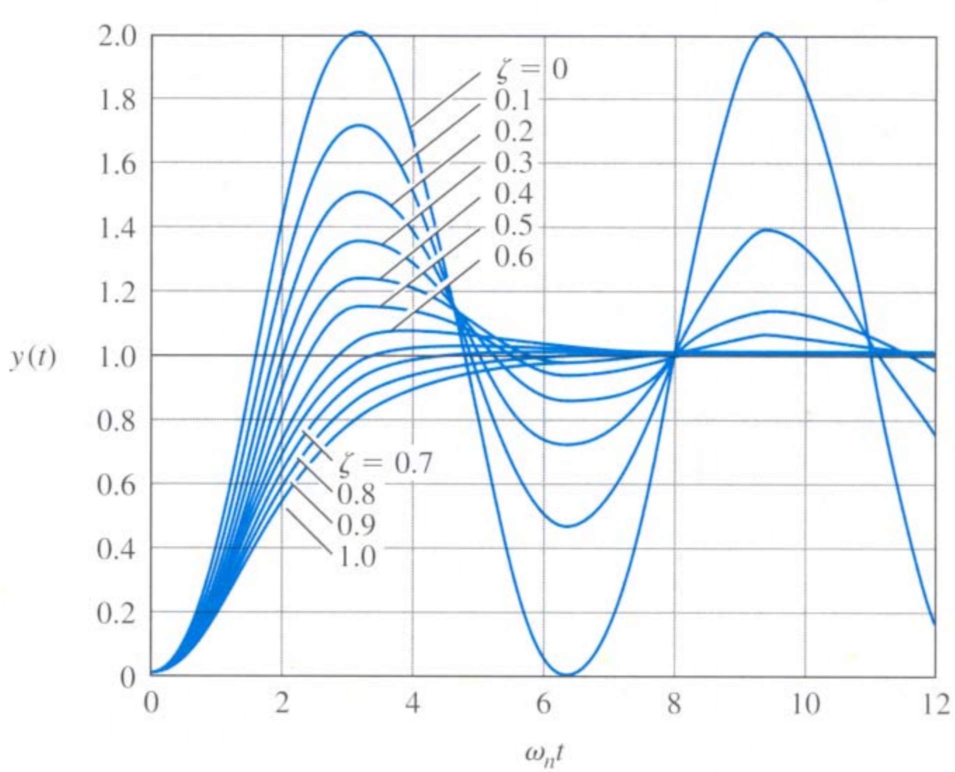
- The damping factor and natural frequency can be computed by

$$\begin{cases} \zeta\omega_n = 1 \\ \omega_n^2 = 1 + k_p \end{cases} \Rightarrow \begin{cases} \omega_n = \sqrt{1+k_p} \\ \zeta = \frac{1}{\sqrt{1+k_p}} \end{cases}$$

Reminder:

$$R(s) = \frac{1}{s} \rightarrow \boxed{H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \rightarrow Y(s)$$

- The behavior of the system is as follows:



The behavior of the system is fully characterized by:

- ζ the dumping factor
- ω_n the natural frequency

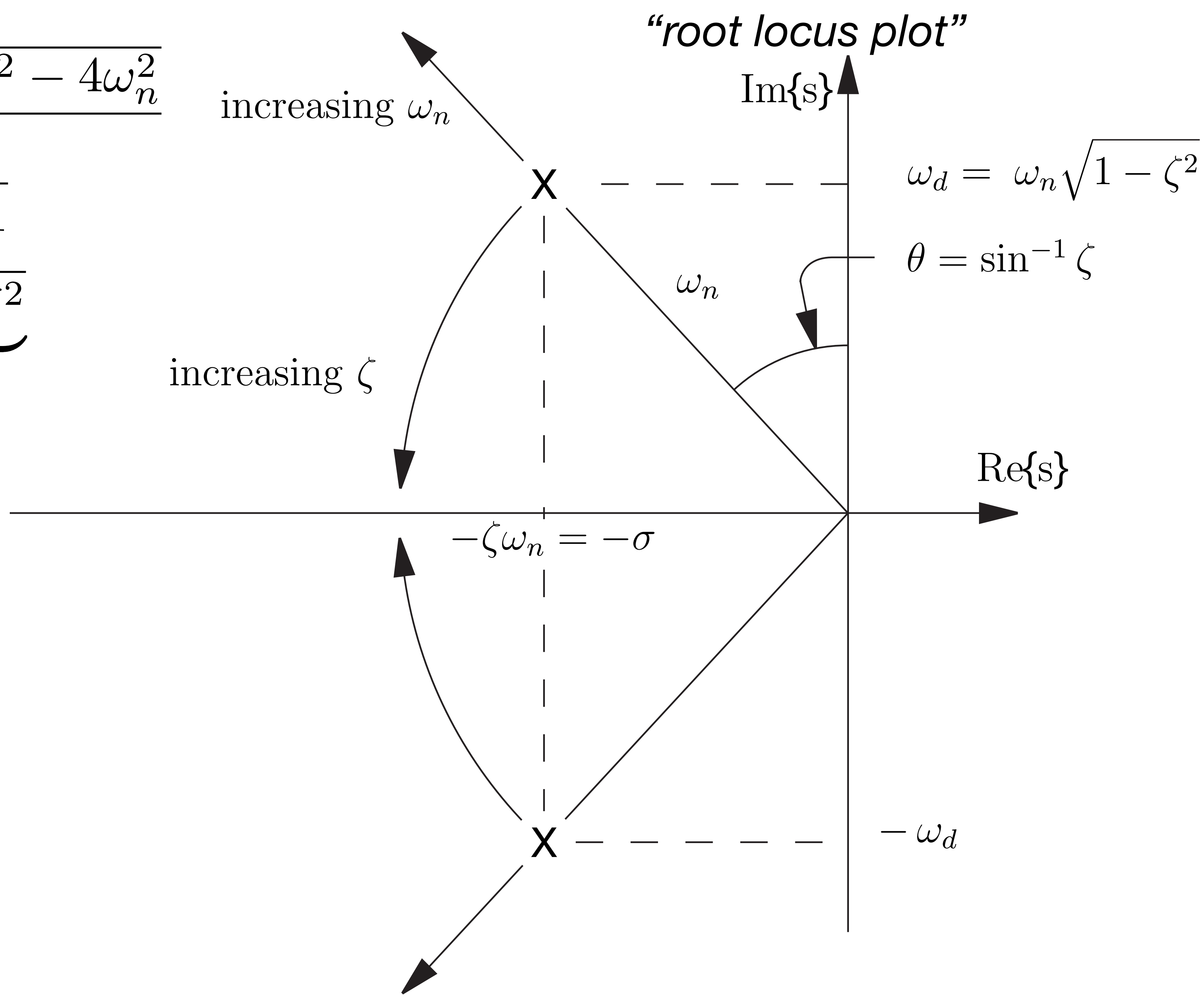
Reminder (continued):

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

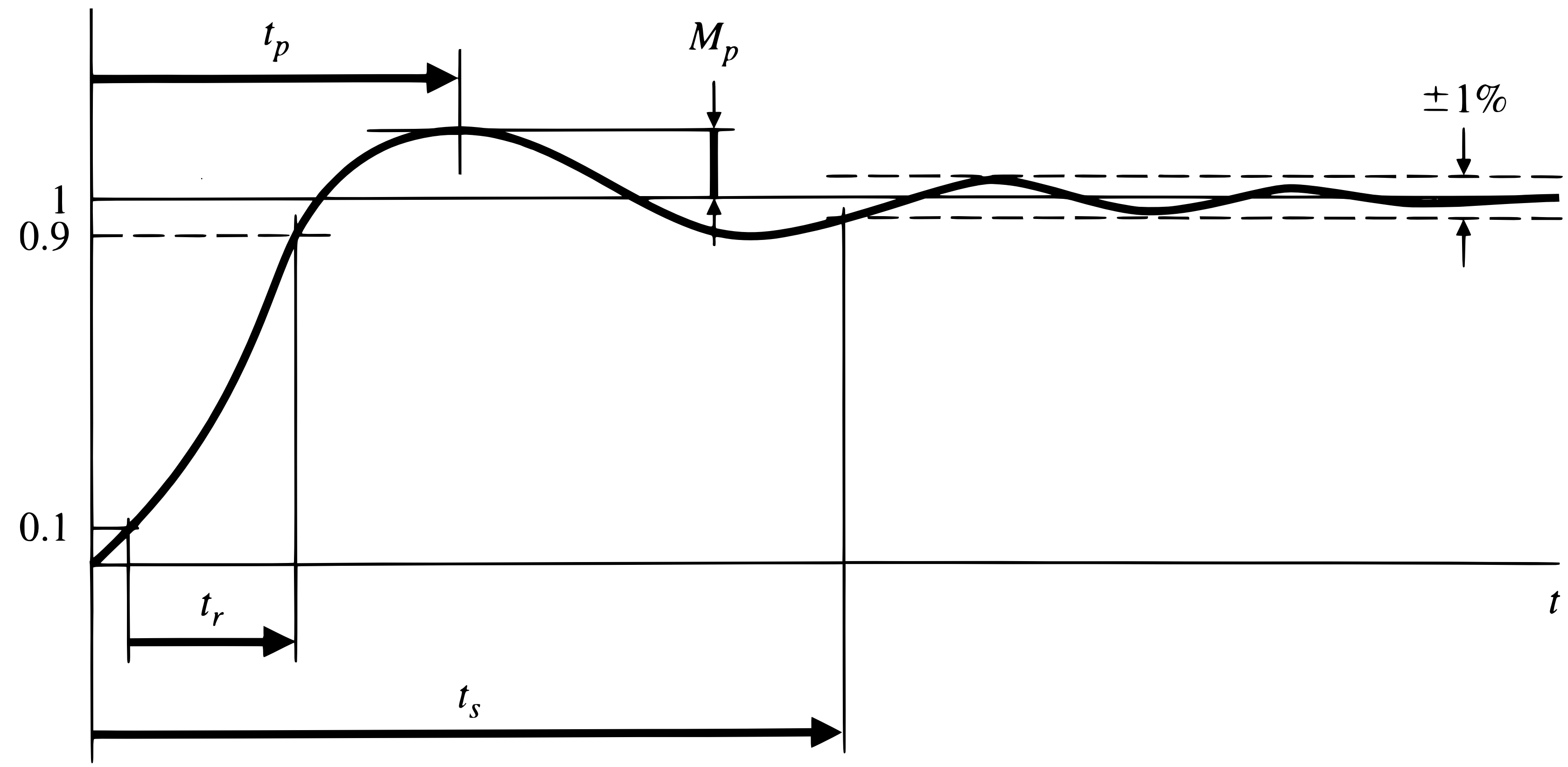
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$= \underbrace{-\zeta\omega_n}_{-\sigma} \pm j\, \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$



Reminder (continued):

- Typical specifications for the step response:



- Steady-state accuracy e_{ss}
- Rise time (10% – 90%) $t_r = 1.8/(\omega_n)$
- Peak overshoot $M_p \approx e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ or $\zeta \geq 0.6\left(1 - \frac{M_p \text{ in } \%}{100}\right)$
- Settling time (1%) $t_s = 4.6/(\zeta\omega_n)$

- Steady-state errors using the Final Value Theorem (FVT):

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s) = \lim_{s\to 0} s\frac{k_p}{s^2+2s+1+k_p}\frac{1}{s}$$
$$= \lim_{s\to 0} \frac{k_p}{s^2+2s+1+k_p} = \frac{k_p}{1+k_p}$$

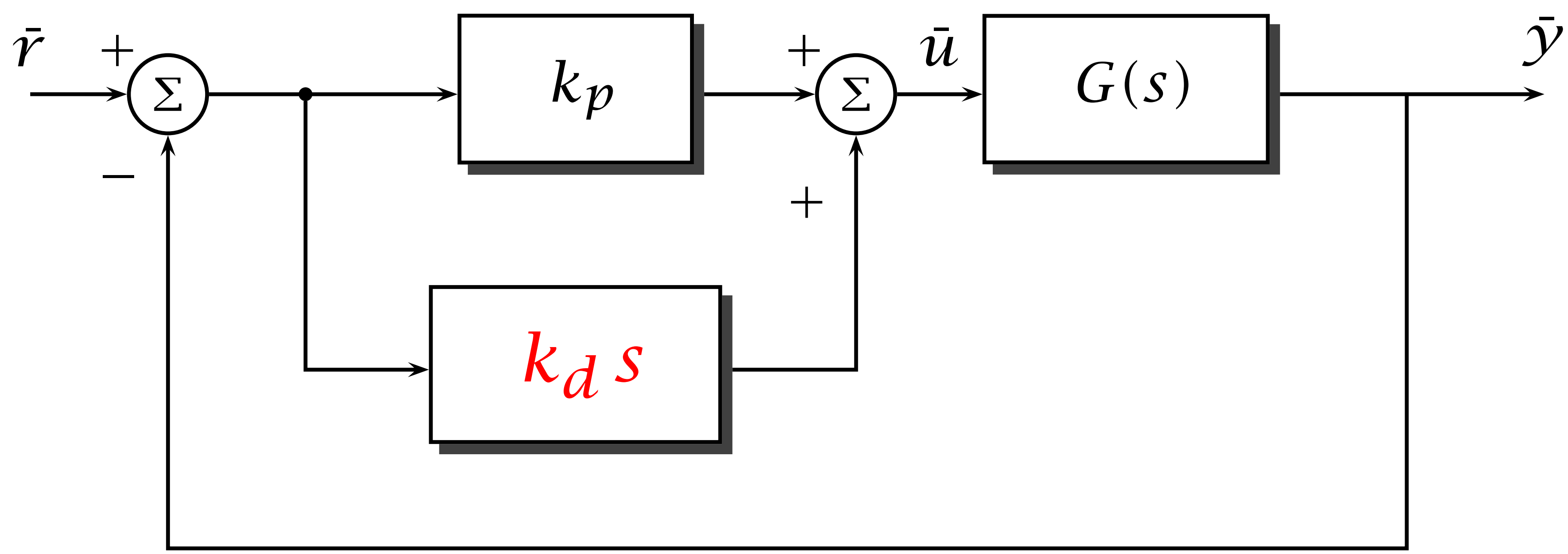
Therefore, the steady-state error is

$$\lim_{t\to\infty} e(t) = 1 - \lim_{t\to\infty} y(t) = \frac{1}{1+k_p}$$

- Hence, in this example, increasing k_p gives
  - ✓ smaller steady-state errors
  - – but a larger and more oscillatory transient response

The root locus concept - Proportional + Derivative control

- In this case, $K(s) = k_p + k_d s$



- Typical result of increasing the gain k_d, (for systems where $G(s)$ is stable):
  - ✓ Increased damping
  - – Greater sensitivity to noise

Example 2:

- Consider the following critically damped 2nd-order system

$$G(s) = \frac{1}{(s+1)^2}$$

Then

$$\bar{y}(s) = \frac{(k_p + k_d s)G(s)}{1 + (k_p + k_d s)G(s)}\bar{r}(s) = \frac{(k_p + k_d s)\frac{1}{(s+1)^2}}{1 + (k_p + k_d s)\frac{1}{(s+1)^2}}\bar{r}(s)$$

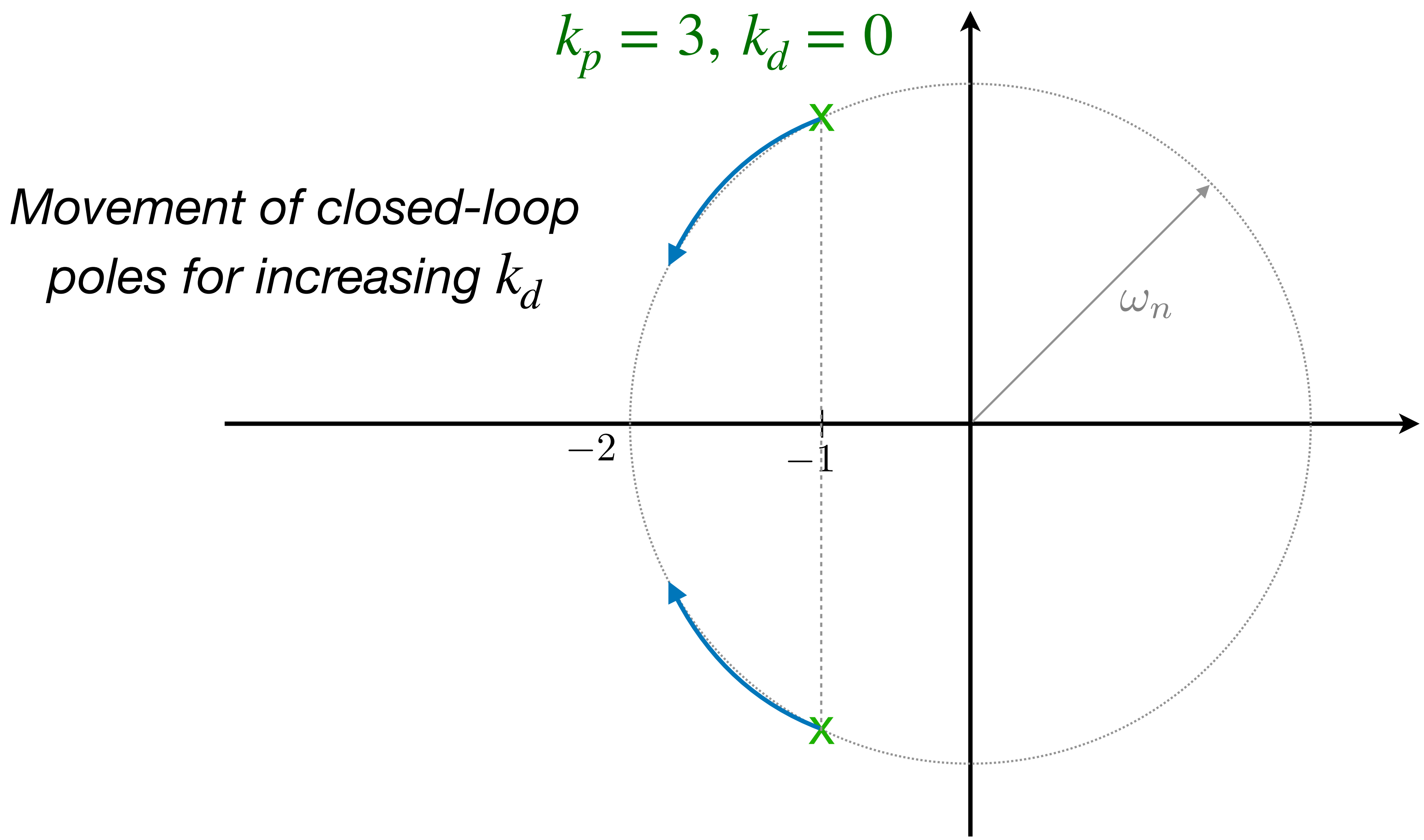
$$= \frac{(k_p + k_d s)}{s^2 + (2 + k_d)s + 1 + k_p}\bar{r}(s)$$

The characteristic equation representing this system is

$$\Delta(s) = s^2 + (2 + k_d)s + 1 + k_p = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

- Solving for the damping factor

$$\begin{cases} 2\zeta\omega_n = 2 + k_d \\ \omega_n^2 = 1 + k_p \end{cases} \Rightarrow \begin{cases} \omega_n = \sqrt{1 + k_p} \\ \zeta = \frac{2+k_d}{2\sqrt{1+k_p}} \end{cases}$$



"root locus plot"

The root locus procedure

- **We will develop an orderly procedure of 6 steps that facilitates the rapid sketching of the root locus**

- **Step 1:** Locate the poles *and zeros* on the s-plane with selected symbols (by convention, we use 'x' to denote poles and 'o' to denote zeros)

  For example, consider the case $1 + k_p G(s) = 0$. We can write it in the form

$$1 + k_p \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} = 0$$

  By rewriting the above equation as

$$\prod_{j=1}^{n}(s - p_j) + k_p \prod_{i=1}^{m}(s - z_i) = 0$$

  it is easy to observe that for $k_p = 0$ the poles of the characteristic equation are the same as the poles of system $G(s)$

Alternatively, the above equation, for $k_p \neq 0$, can be written as

$$\frac{1}{k_p}\prod_{j=1}^{n}(s-p_j)+\prod_{i=1}^{m}(s-z_i)=0$$

Therefore, as $k_p \to \infty$ the roots of the characteristic equation are the zeros of system $G(s)$

Therefore,

- the locus of the roots of the characteristic equation $1 + k_p G(s) = 0$ begins at the poles of $G(s)$ and ends at the zeros of $G(s)$ as k_p increases from zero to infinity
- For most systems $G(s)$ that we will encounter, several of the zeros of $G(s)$ lie at infinity in the s-plane.This is because most of our systems have more poles than zeros. With n poles and m zeros and $n > m$, we have $n - m$ branches of the root locus approaching the $n - m$ zeros at infinity

- **Step 2:** Locate the segments of the real axis that are root loci. *The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.*

This can be observed by the phase requirement: $\angle(k_p G(s)) = 180^o + k360^o$. Therefore,

$$\sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_i) = 180^o + k360^o$$

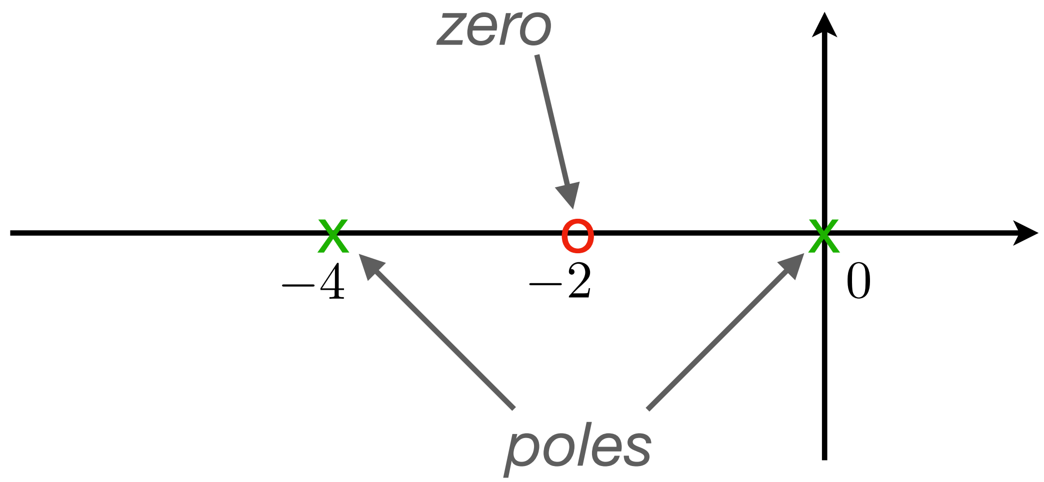
Only to the left of an odd number and zeros the equation above is satisfied on the real axis!

Example 3:

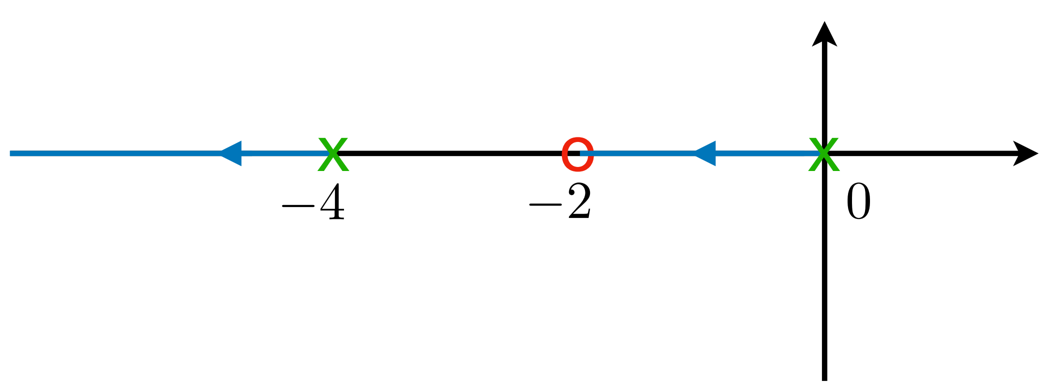
- A single-loop feedback control system has the characteristic equation

$$1 + k_p G(s) = 1 + k_p \frac{2(s+2)}{s(s+4)} = 0$$

Step 1: The zeros and the poles of $G(s)$ are shown in the figure below



Step 2: Due to the angle criterion, the locus begins at the pole and ends at the zeros, and therefore the locus of roots appears as below



Observations:

- Since the loci begin at the poles and end at the zeros, the number of separate loci is equal to the number of poles since the number of poles is greater than or equal to the number of zeros
- The root loci must be symmetrical with respect to the horizontal real axis because the complex roots must appear as pairs of complex conjugate roots
- Since the system in this example has two real poles and one real zero, the second locus segment ends at a zero at negative infinity
- To evaluate the gain k_p at a specific root location on the locus, we use the magnitude criterion. For example, to have a root at $s_1 = -1$, we have

$$k_p \frac{2|s_1 + 2|}{|s_1||s_1 + 4|} = 1$$

$$k_p \frac{2|-1+2|}{|-1||-1+4|} = 1$$

$$k_p \frac{2}{3} = 1 \Rightarrow k_p = \frac{3}{2}$$

- **Step 3:** The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A. When the number of finite zeros of $G(s)$, m, is less than the number of poles, n, by the number $N = n - m$, then N sections of loci proceed to the zeros at infinity along asymptotes as k_p approaches infinity.

The *asymptotes are centered at a point (asymptote centroid) on the real axis* given by

$$\sigma_A = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m}$$

The *angle of the asymptotes with respect to the real axis* is

$$\phi_A = \frac{(2k+1)180^o}{n - m}, \quad k = 0, 1, 2, \ldots, n - m - 1$$

(proof can be found in reference book, pp. 452-453)

Example 4:

- A single-loop feedback control system has the characteristic equation

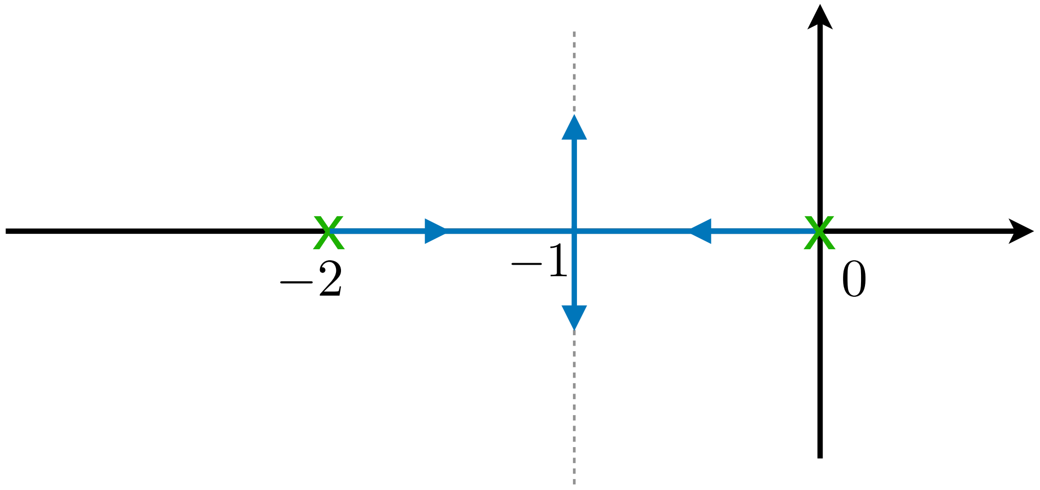
$$1 + k_p G(s) = 1 + k_p \frac{1}{s(s+2)} = 0$$

Step 3: Since $n - m = 2$, we expect two loci to end at zeros at infinity. The asymptotes of the loci are located at a center

$$\sigma_A = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} = \frac{0 + (-2)}{2} = -1$$

and at angles

$$\phi_A = \frac{(2k+1)180^o}{n-m} = \frac{(2k+1)180^o}{2} = (2k+1)90^o, \quad k = 0, 1$$

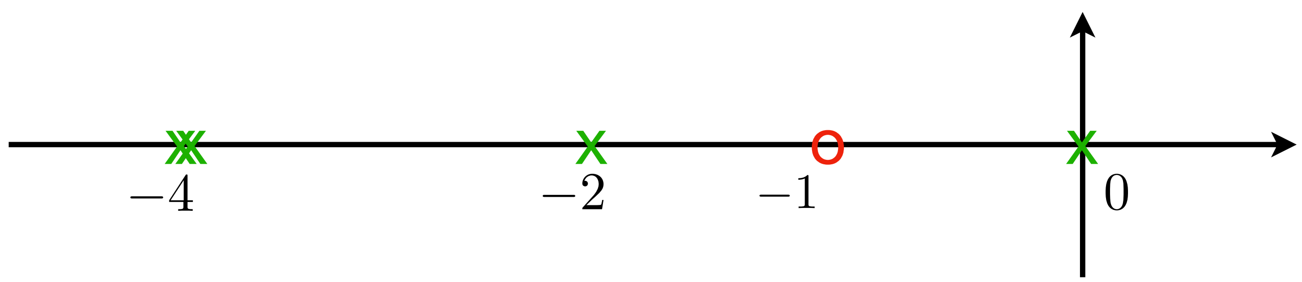


Example 5:

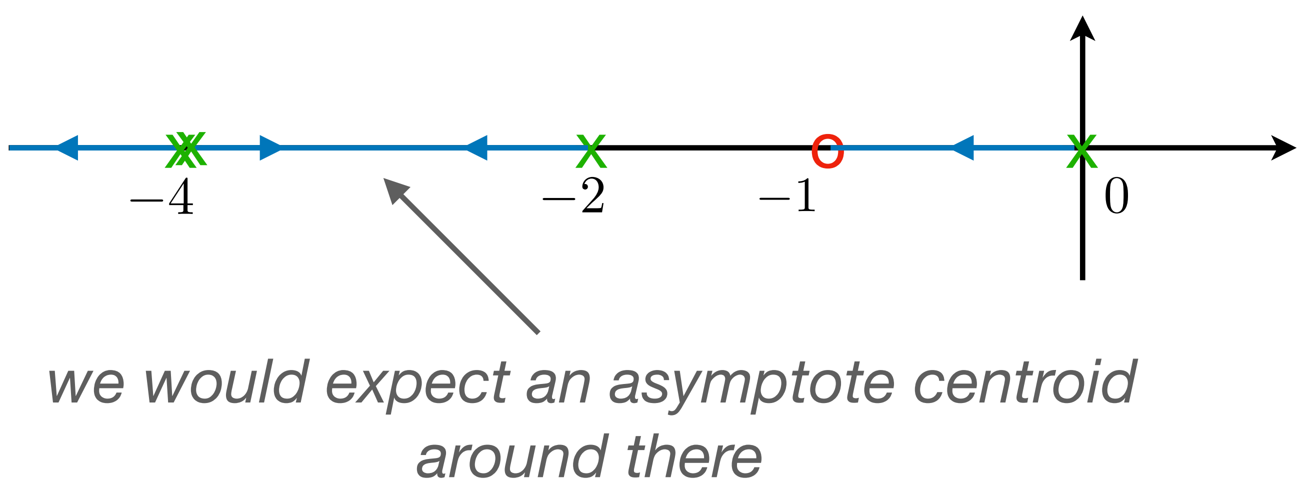
- A single-loop feedback control system has the characteristic equation

$$1 + k_p G(s) = 1 + k_p \frac{s+1}{s(s+2)(s+4)^2} = 0$$

Step 1: Locate the poles and zeros:



Step 2: Locate the segments of the real axis that are root loci:



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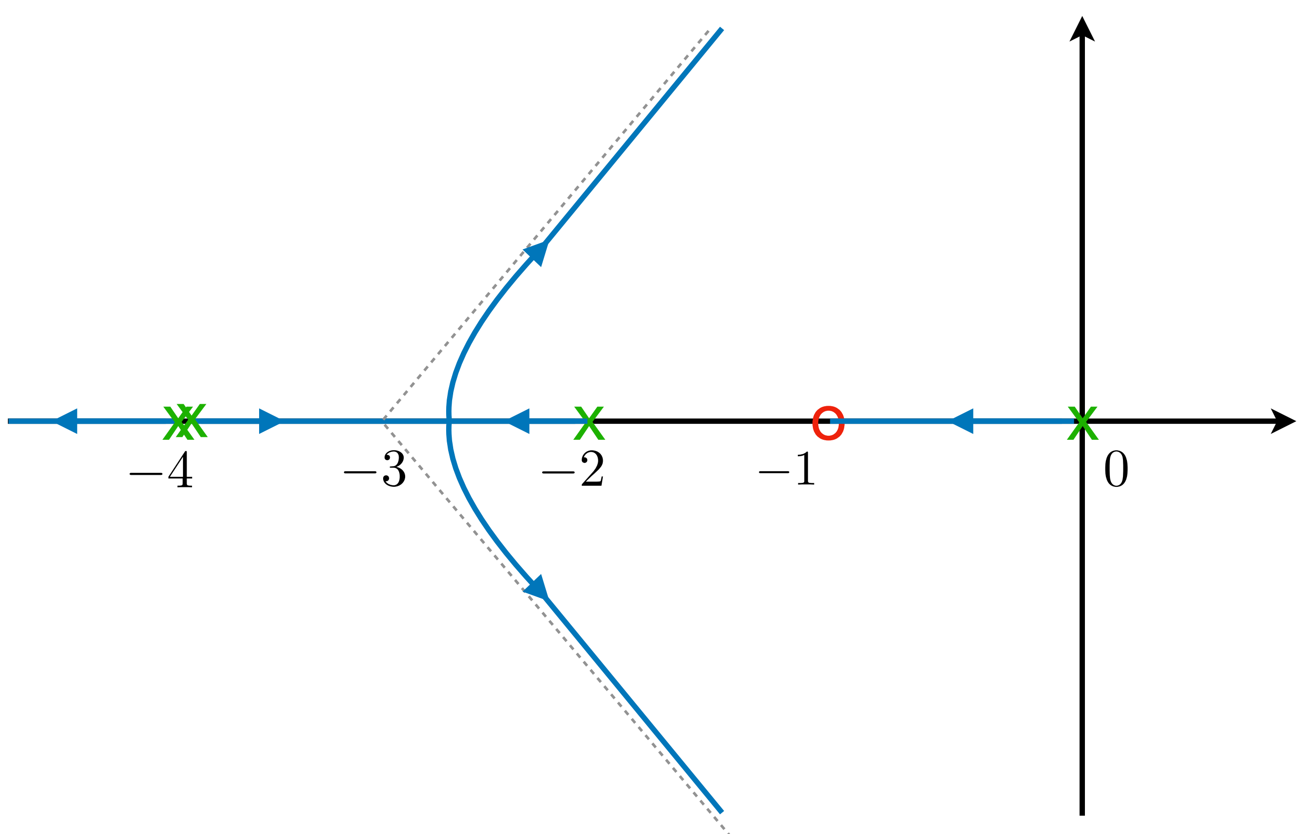
Step 3: Since $n - m = 3$, we expect 3 loci to end at zeros at infinity. The asymptotes of the loci are located at a center

$$\sigma_A = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} = \frac{0 + (-2) + 2(-4) - (-1)}{3} = \frac{-9}{3} = -3$$

and at angles

$$\phi_A = \frac{(2k+1)180^o}{n-m} = \frac{(2k+1)180^o}{3} = (2k+1)60^o, \quad k = 0, 1, 2$$

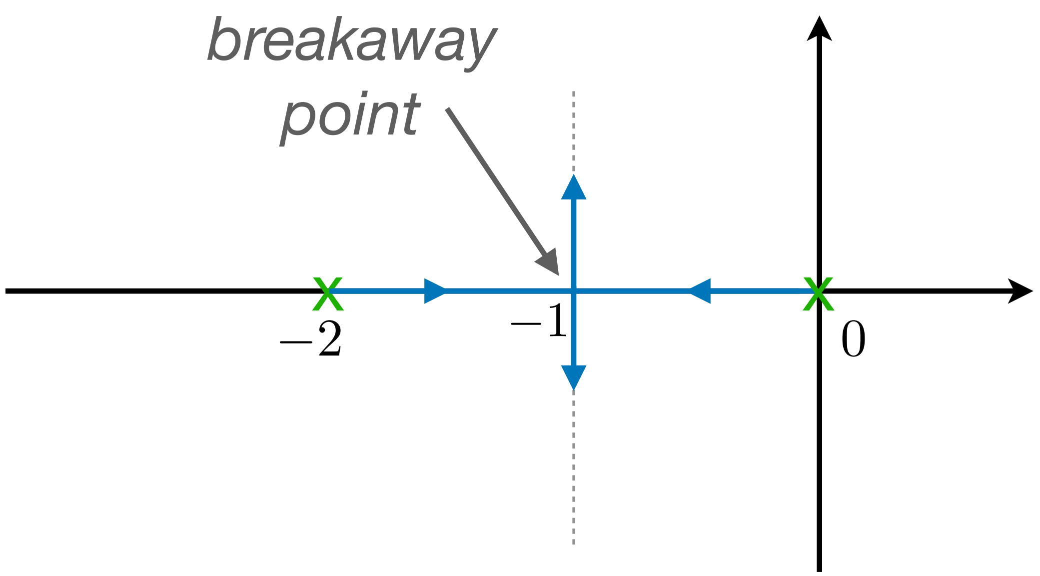


- **Step 4:** Determine the *break-in* and *breakaway* points on the real axis (if any). For each $s = \sigma$ on a real-axis segment of the root locus,

$$1 + k_p G(\sigma) = 0 \Rightarrow k_p = -\frac{1}{G(\sigma)}$$

Real-axis break-in and breakaway points are the real values of σ for which

$$\frac{dk_p(\sigma)}{d\sigma} = 0$$



Alternatively, it can be shown that a breakaway or break-in point satisfy

$$\sum_i \frac{1}{\sigma - z_i} = \sum_j \frac{1}{\sigma - p_j}$$

Proof: Consider the characteristic equation

$$1 + k_p G(s) = 0 \Rightarrow 1 + k_p \frac{Y(s)}{X(s)} = 0 \Rightarrow X(s) + k_p Y(s) = 0$$

For a small change in k_p, we have

$$X(s) + (k_p + \delta k_p) Y(s) = 0 \Rightarrow 1 + \frac{\delta k_p Y(s)}{X(s) + k_p Y(s)} = 0$$

Since the denominator is the original characteristic equation, a multiplicity r of roots exists at a break-in or a breakaway point. Hence,

$$\frac{Y(s)}{X(s) + k_p Y(s)} = \frac{C_i}{(s - s_i)^r} = \frac{C_i}{(\delta s)^r}$$

Therefore,

$$1 + \delta k_p \frac{C_i}{(\delta s)^r} = 0 \Rightarrow \frac{\delta k_p}{\delta s} = -\frac{(\delta s)^{r-1}}{C_i}$$

As we allow $\delta k_p \rightarrow 0$, we obtain

$$\frac{dk_p}{ds} = 0$$

Example 6:

- A single-loop feedback control system has the characteristic equation

$$1 + k_pG(s) = 1 + k_p\frac{(s-3)(s-5)}{(s+1)(s+2)} = 0$$

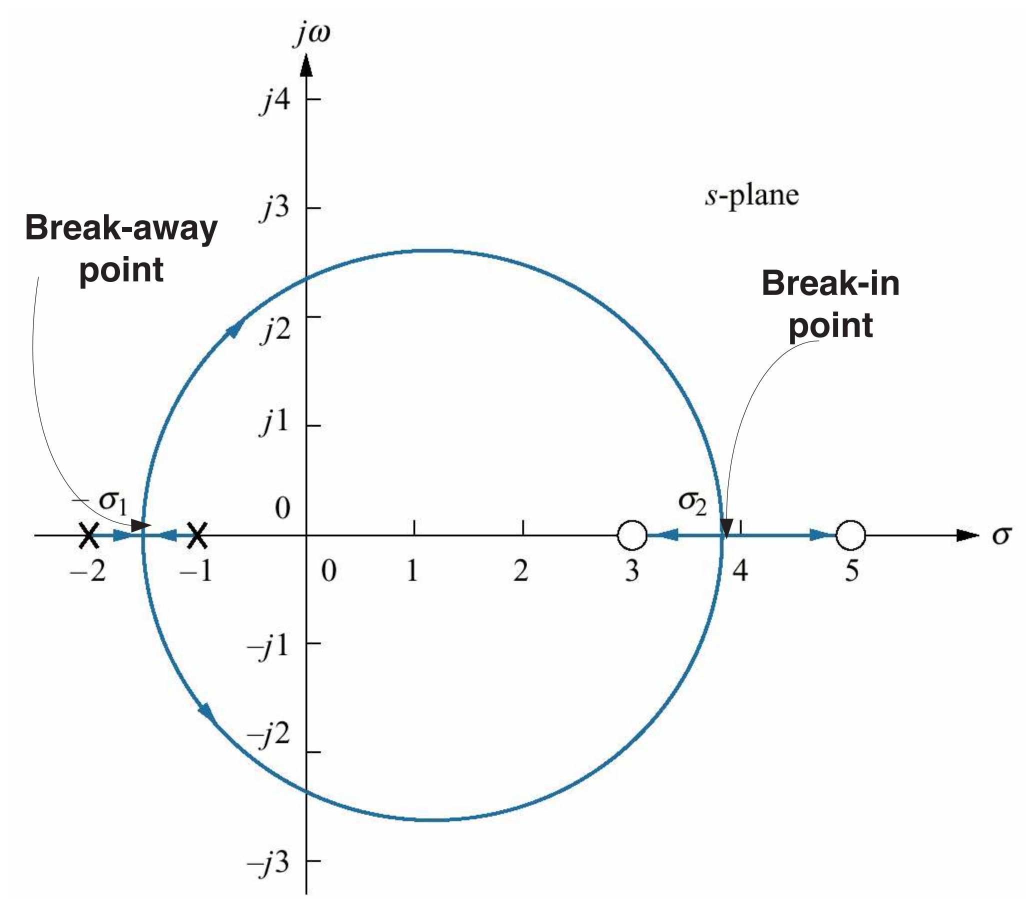
So, on the real-axis segments we have

$$k_p(\sigma) = -\frac{(\sigma+1)(\sigma+2)}{(\sigma-3)(\sigma-5)} = -\frac{\sigma^2+3\sigma+2}{\sigma^2-8\sigma+15}$$

Taking the derivative with respect to σ,

$$\frac{dk_p(\sigma)}{d\sigma} = -\frac{11\sigma^2-26\sigma-61}{(\sigma^2-8\sigma+15)^2} = 0$$

$$\Rightarrow \begin{cases} \sigma_1 = -1.45 \\ \sigma_2 = 3.82 \end{cases}$$



- **Step 5:** Determine where the locus crosses the imaginary axis (if it does so).

1st way: Using the Routh-Hurwitz criterion:

  - When we have $j\omega$-axis crossings, the Routh-table has all zeros at a row
  - Find the value of k_p for which a row of zeros is achieved in the Routh-table.

2nd way: Alternatively, If $s = j\omega$ is a closed-loop pole on the imaginary axis, then

$$1 + k_p G(j\omega) = 0$$

The real and imaginary parts of the equation above provide us with 2 equations with two unknowns k_p and ω (i.e., the critical gain beyond which the system goes unstable, and the oscillation frequency at the critical gain)

Example 7 (1st way):

- A single-loop feedback control system with closed-loop transfer function

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

The Routh table is given by

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\frac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

The row s^1 is zero for $K = 9.65$. For this K, the previous row polynomial is

$$(90-K)s^2 + 21K = 0$$

whose roots are $s = \pm j1.59$

Example 7 (2nd way):

- A single-loop feedback control system has the characteristic equation

$$1 + k_p G(s) = 1 + k_p \frac{s+3}{s(s+1)(s+2)(s+4)} = 0$$

So, on the imaginary-axis we have

$$k_p G(j\omega) = -1 \Rightarrow -\omega^4 + j7\omega^3 + 14\omega^2 - j(k_p + 8)\omega - 3K = 0$$

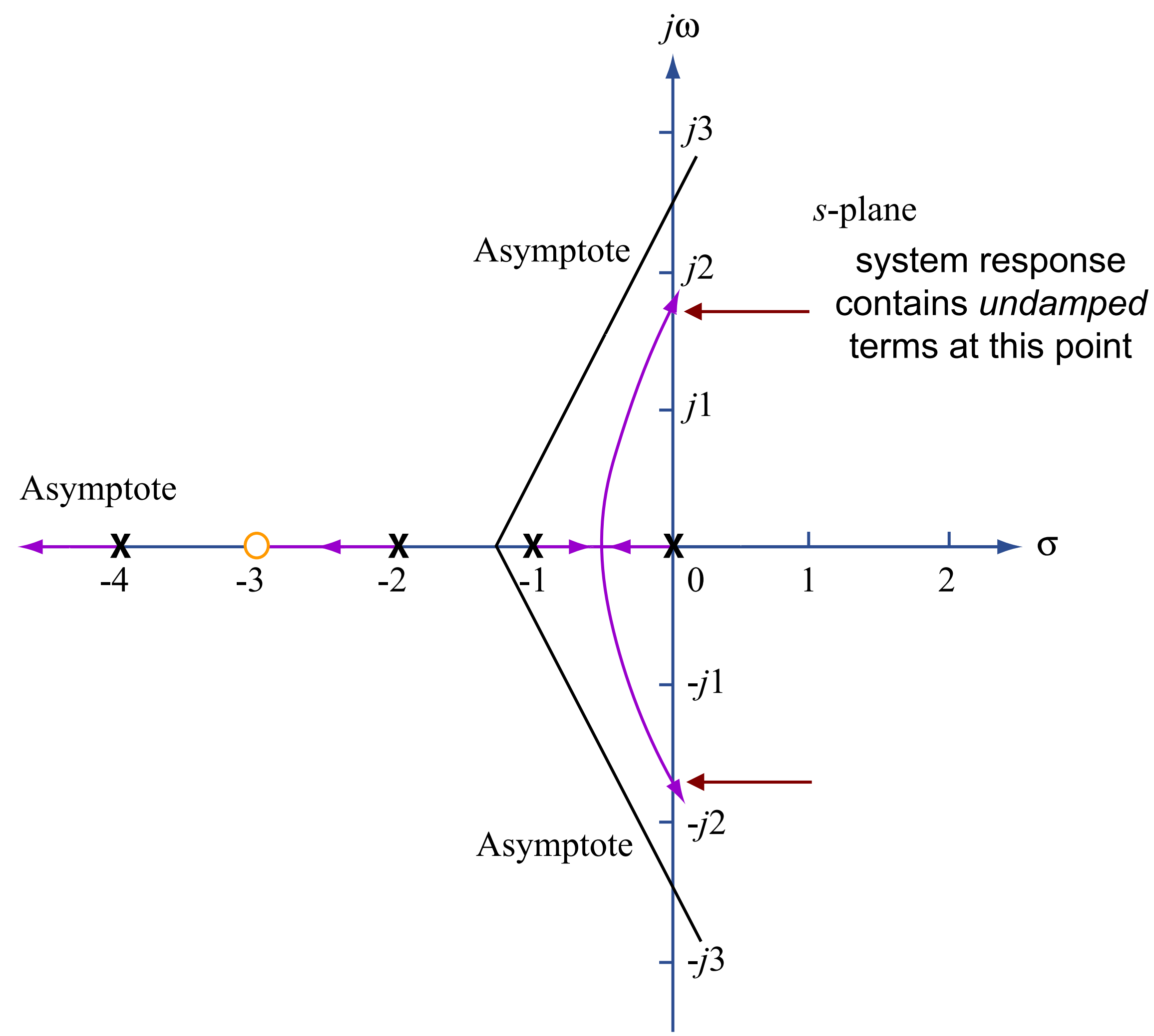
Separating real and imaginary parts,

$$\begin{cases} -\omega^4 + 14\omega^2 - 3K = 0 \\ 7\omega^3 - (k_p + 8)\omega = 0 \end{cases}$$

In the second equation, we can discard the trivial solution $\omega = 0$. It then yields

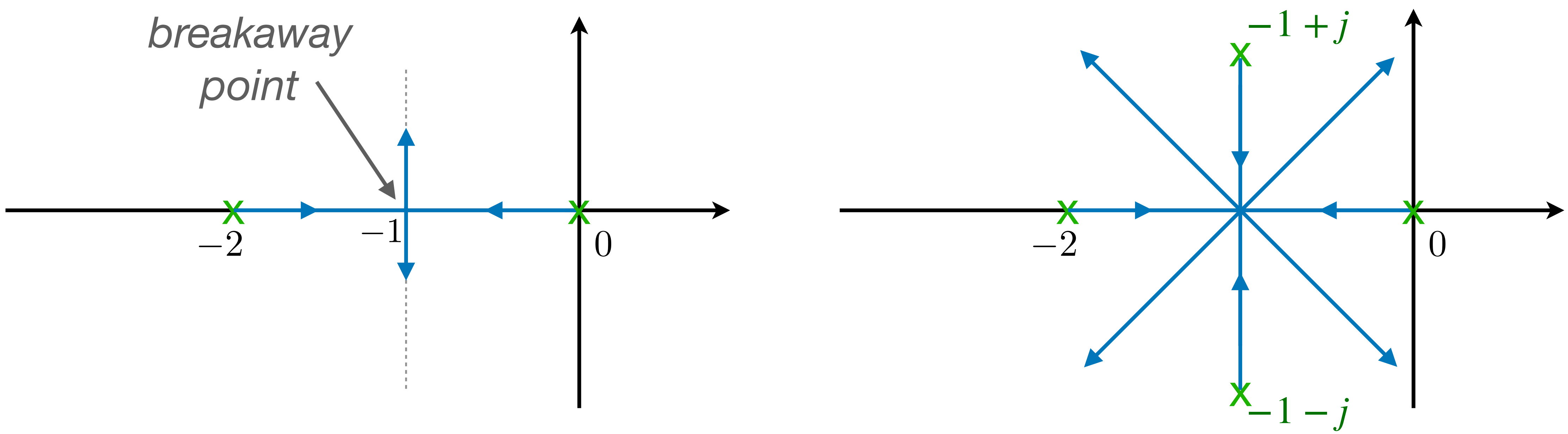
$$\omega^2 = \frac{k_p + 8}{7}$$

And substitute in the first to find k_p



- **Step 6:** Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero, using the phase angle criterion.

  The angle of locus departure from a pole is the difference between the net angle due to all other poles and zeros and the criterion angle of $\pm(2k+1)180^o$, and similarly for the locus angle of arrival at a zero.

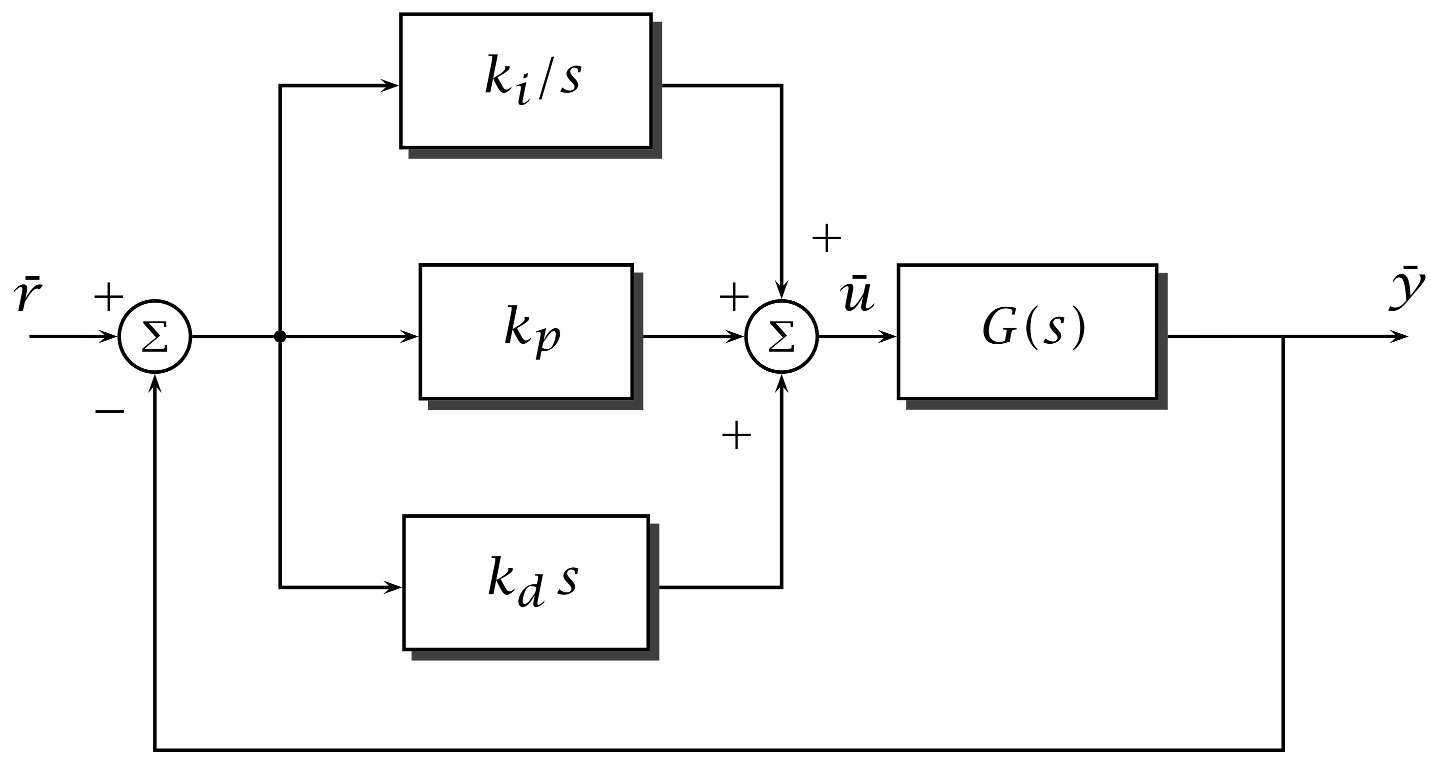


Due to the phase criterion, the tangents to the loci at the breakaway point are equally spaced over 360°. Therefore,

- in left figure, the two loci at the breakaway point are spaced 180° apart
- in right figure, the four loci are spaced 90° apart

Proportional + Integral + Derivative control

- In this case, $K(s) = k_p + k_i/s + k_p s$



- The continuous-time PID controller (in time domain) is

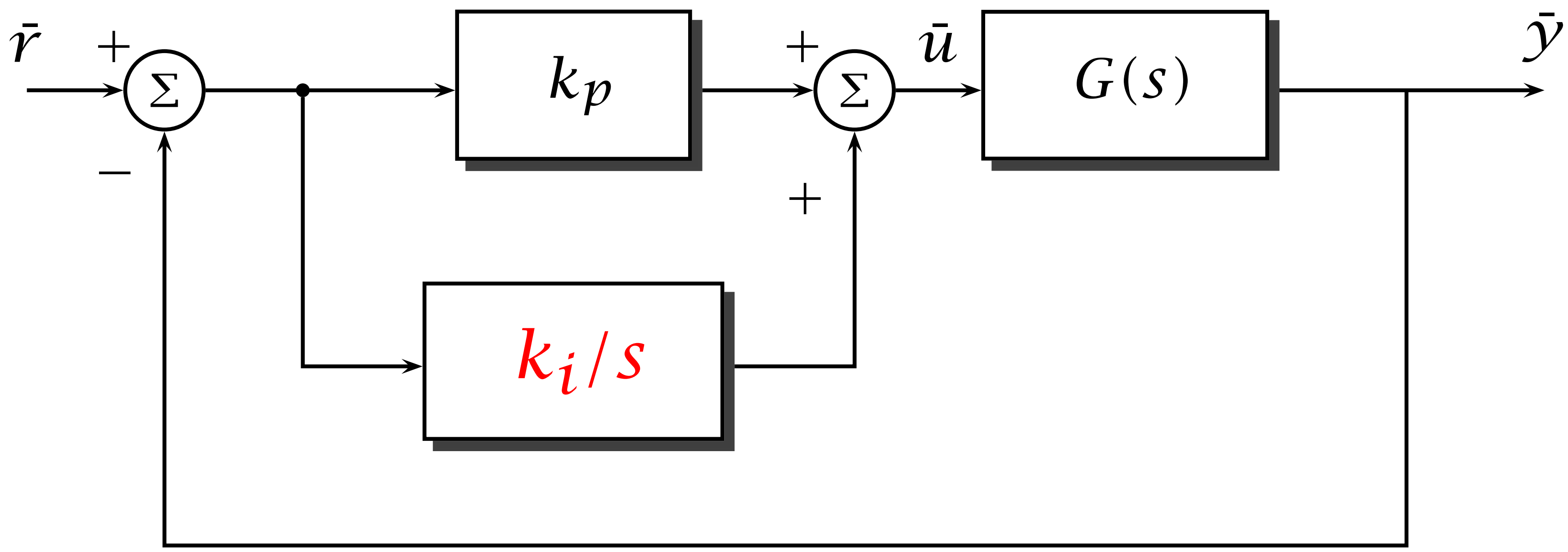
$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de(t)}{dt}$$

- Taking Laplace transforms:

$$\bar{u}(s) = \left(K_p + \frac{K_i}{s} + K_d s\right)\bar{e}(s)$$

Proportional + Integral control

- In this case, $K(s) = k_p + k_i/s$



- Typical result of having integral control k_i:
  ✓ (if stabilizing) always results in zero steady-state error, in the presence of constant disturbances and demands

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Example 8:

- Consider the following critically damped 2nd-order system

$$G(s) = \frac{1}{(s+1)^2}$$

Then

$$\bar{y}(s) = \frac{(k_p + k_i/s)G(s)}{1 + (k_p + k_i/s)G(s)}\bar{r}(s) = \frac{(k_p + k_i/s)\frac{1}{(s+1)^2}}{1 + (k_p + k_i/s)\frac{1}{(s+1)^2}}\bar{r}(s)$$

$$= \frac{(k_p s + k_i)}{s(s+1)^2 + k_p s + k_i}\bar{r}(s)$$

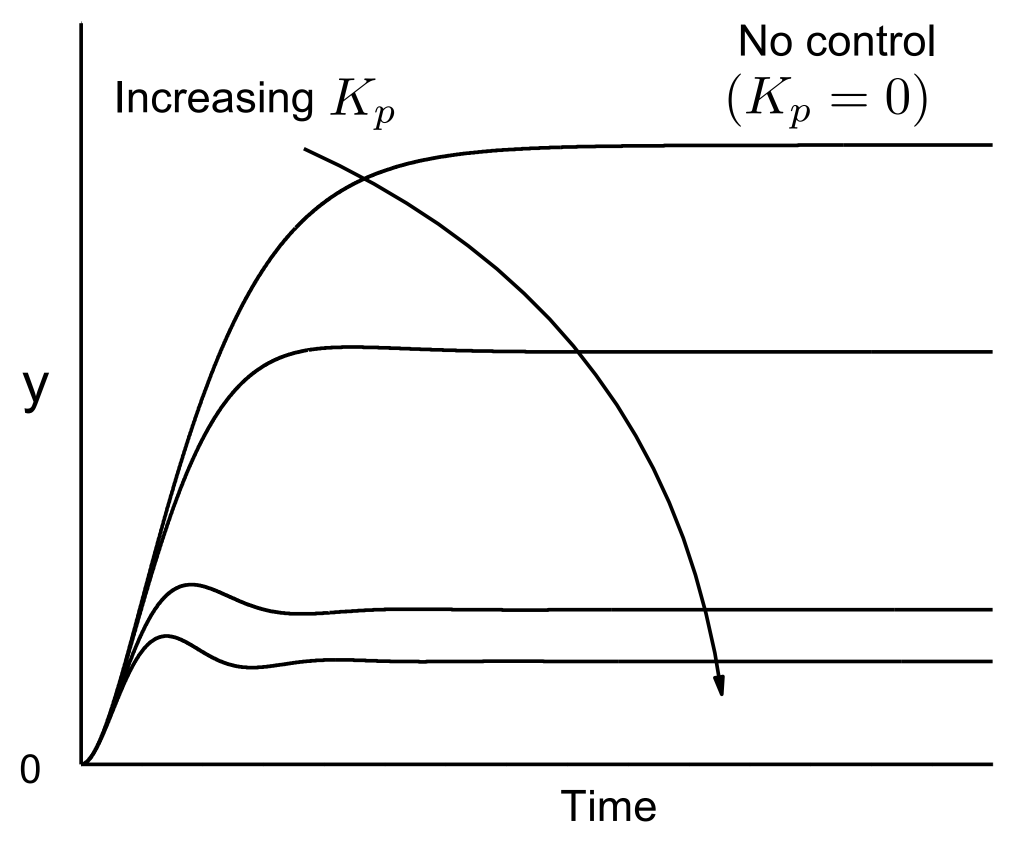
Applying the FVT:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s) = \lim_{s\to 0} s\frac{(k_p s + k_i)}{s(s+1)^2 + k_p s + k_i}\frac{1}{s} = 1$$

Therefore, there is no steady-state error!

P-Controller

- The obvious method - proportional control
- This method fails if, for instance, the error corresponds to more than a single task or the system changes; hence, for the same error, different gains are needed.

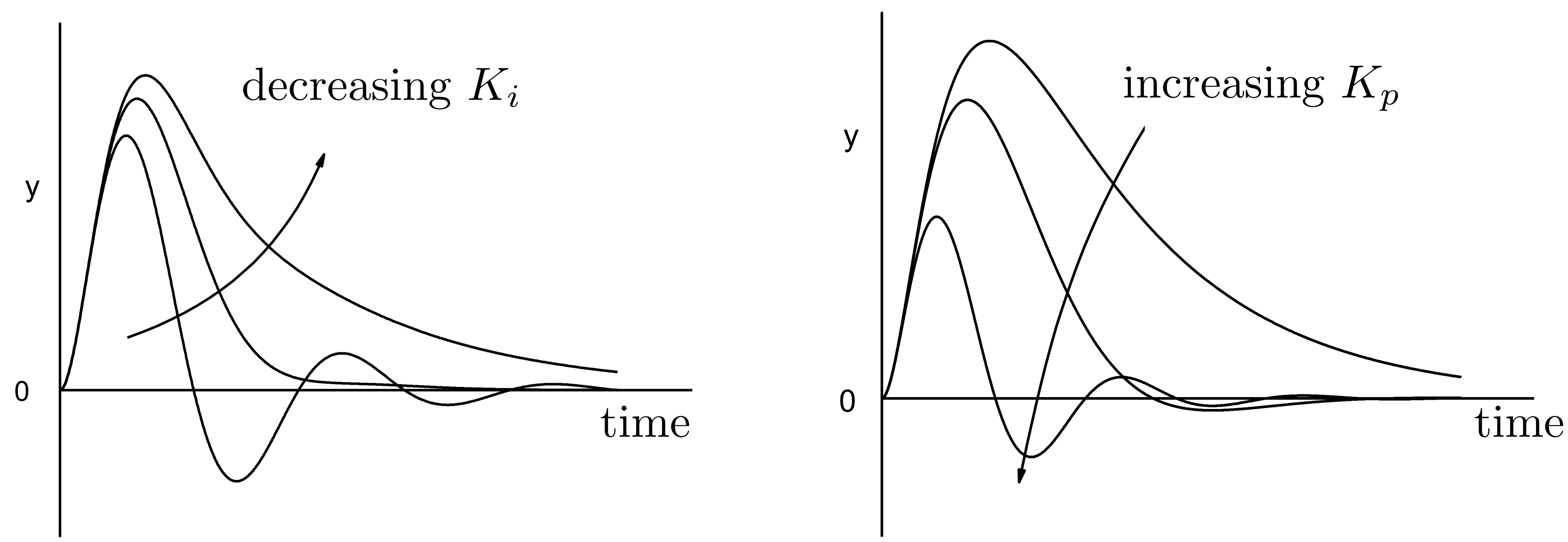


- That's where the integral and derivative terms play their part.

I-Controller

- An **integral** term increases action in relation not only to the error, but also the time for which it has persisted. So, if applied control action is not enough to bring the error to zero, this control action will be increased as time passes.

- A pure "**I**" controller could bring the error to zero, however, it would be both slow reacting at the start, brutal, and slow to end, prompting overshoot and oscillations.



- Alternative formulation: change the error in small persistent steps - over time the steps accumulate and add up dependent on past errors;
this is the discrete-time equivalent to integration.

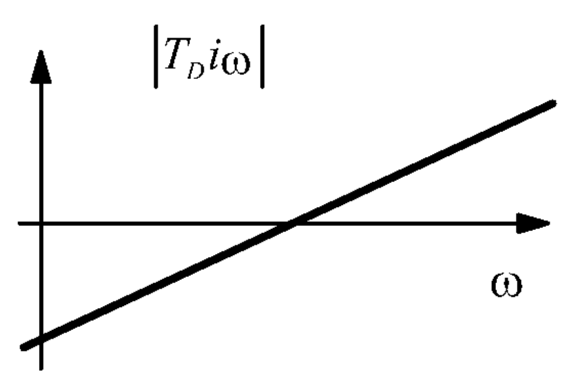
D-Controller

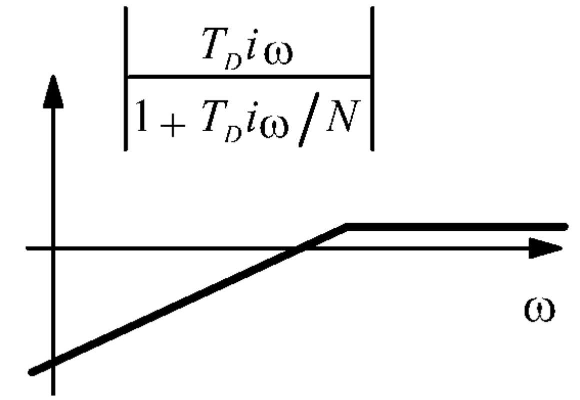
- Aims at flattening the error trajectory into a horizontal line, damping the control applied, and so reduces overshoot

- Ideal derivative control cannot (and must not) be realized in a PID-controller. Practical systems always contain high frequency disturbances (e.g., white noise), which are amplified by derivative control.

- Because of that a lag term is usually added to the derivation.

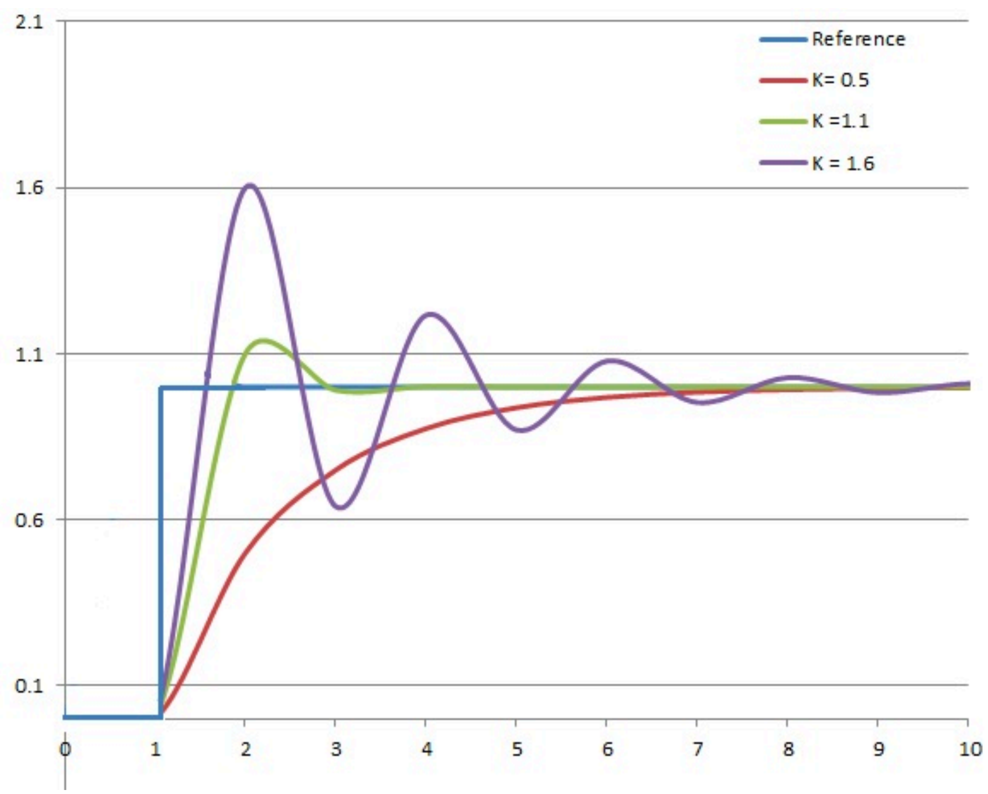
$$K_d s \rightsquigarrow \frac{K_d s}{1 + K_d s/N}$$



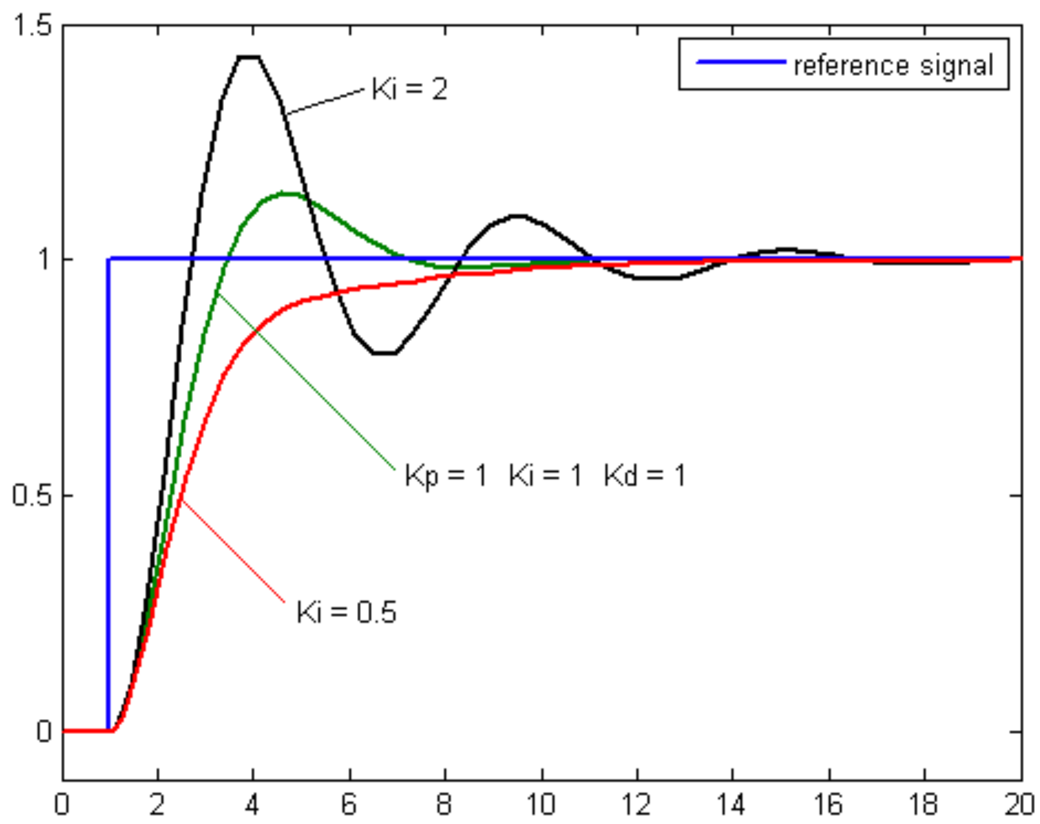


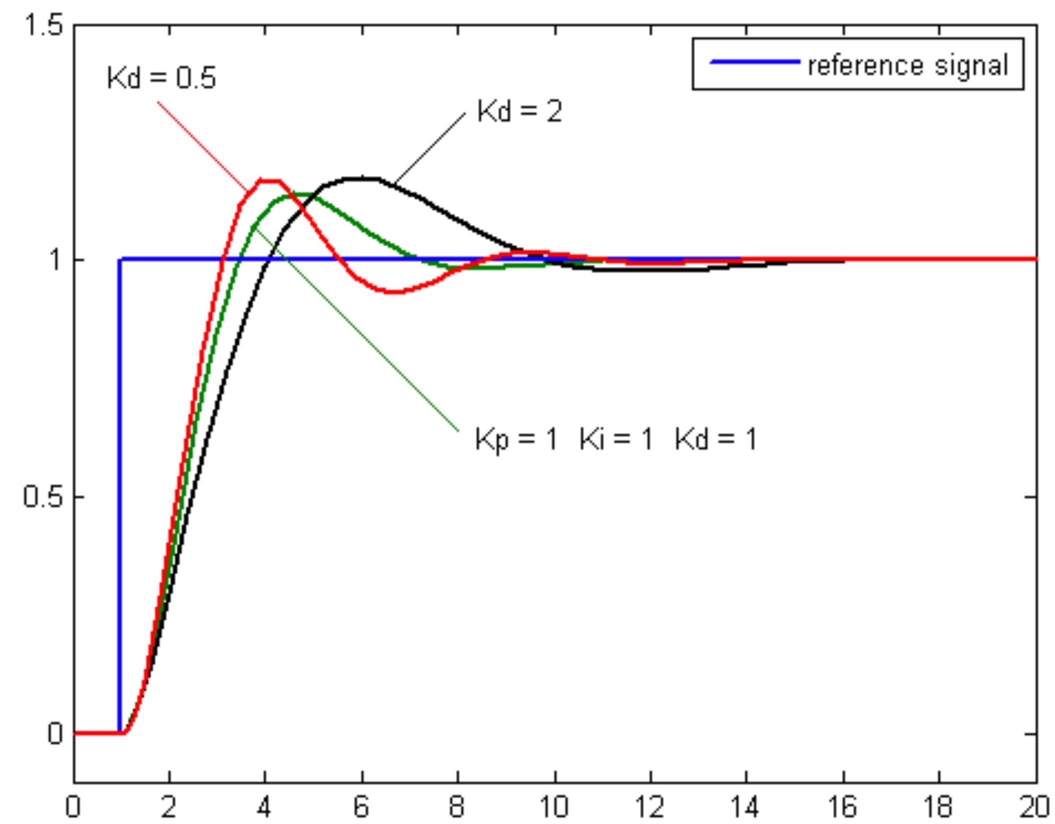
- Other practical modification is to *derivate only the output* (not the reference, not the error signal)

PID-Controller



- **Top-left:** P-controller effect (ID-controllers kept constant)
- **Bottom-left:** I-controller effect (PD-controllers kept constant)
- **Bottom-right:** D-controller effect (PI-controllers kept constant)





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Tuning PID controllers

- The structure of the used discrete PID algorithm must always be told together with the tuning parameters k_p, k_i, k_d.

- Controller design is based on heuristic design methods for selecting the controller parameters.

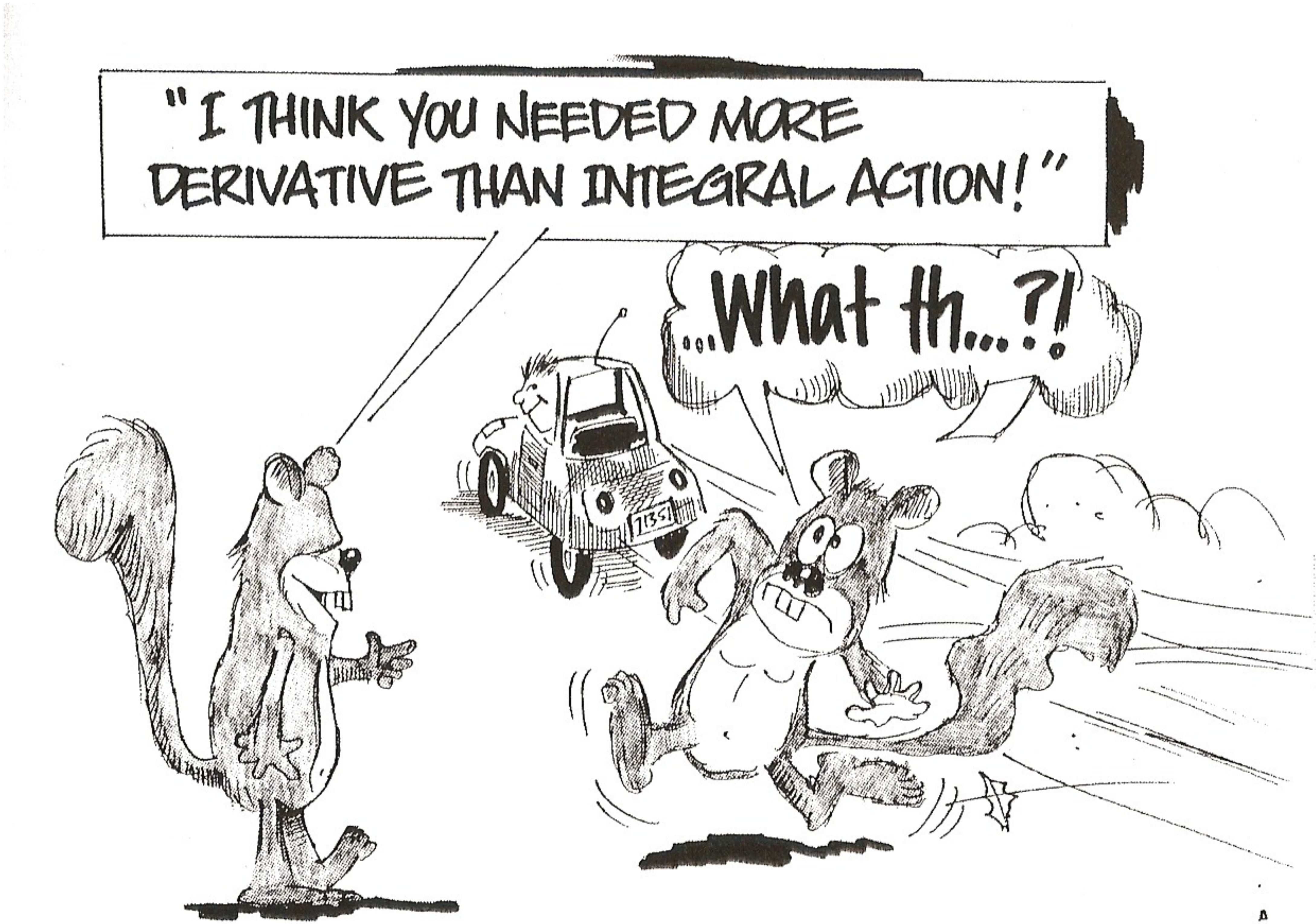
- The principal design goal is stability: The system is stable when the closed loop poles are on the left-half of s-plane

- Secondary criteria are, for example, rise, overshoot, settling time, and steady state error. These can be analyzed graphically from impulse, step and ramp responses of the close loop system

Effects of *increasing* a parameter independently

Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor change	Decrease	Decrease	No effect in theory	Improve if small

PID-Controllers



Learning outcomes

By the end of *this* lecture, you should be able to:

- Understand the concept of the root locus and its role in control system design
- Know how to obtain a root locus plot by sketching or using MATLAB
- Be familiar with the PID controller as a key element of many feedback systems