MEC-E8001

Finite Element Analysis

2023

WEEK 7: THERMO-MECHANICAL ANALYSIS

6 THERMO-MECHANICAL ANALYSIS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on thermo-mechanical FEA:

- ☐ Balance laws and constitutive equations of isotropic thermo-mechanics

- ☐ Stationary thermo-mechanical FEA with solid, plate, and beam elements

- ☐ Virtual work densities of solid, plate, and beam models

MULTIPHYSICS FEA

Multiphysics simulation employs temperature, water contents, etc. with additional balance laws and constitutive equations to predict displacement, temperature, concentration etc. under complex interactions. *A thermo-mechanical model* considers the effect of temperature on mechanical behavior:

- As an unwanted mechanical effect, pipelines and continuous welded rails may bend or buckle in a hot summer.

- Press fit take advantage of thermal expansion and contraction: enveloping parts are assembled into position while hot, then allowed to cool and contract back to their former size. Loosening of a jar lid under heating is based on the opposite mechanism.

- Temperature changes may induce very large stresses.

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume.

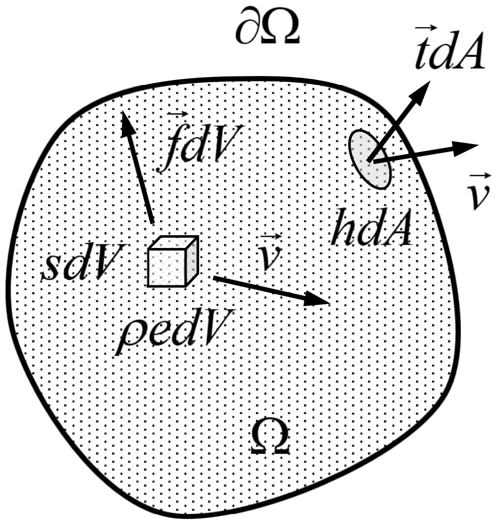
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume.

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

BALANCE OF ENERGY

The rate of change of kinetic and internal energies equals the powers of external forces and added heat, i.e., $\dot{U} + \dot{T} = P_W + P_Q$ where



Internal energy $U = \int_\Omega \rho e dV$

Kinetic energy $T = \int_\Omega \frac{1}{2}\rho\vec{v}\cdot\vec{v}dV$

Power of forces $P_W = \int_\Omega \vec{f}\cdot\vec{v}dV + \int_{\partial\Omega} \vec{t}\cdot\vec{v}dA$

Power of heat $P_Q = \int_\Omega sdV + \int_{\partial\Omega} hdA$

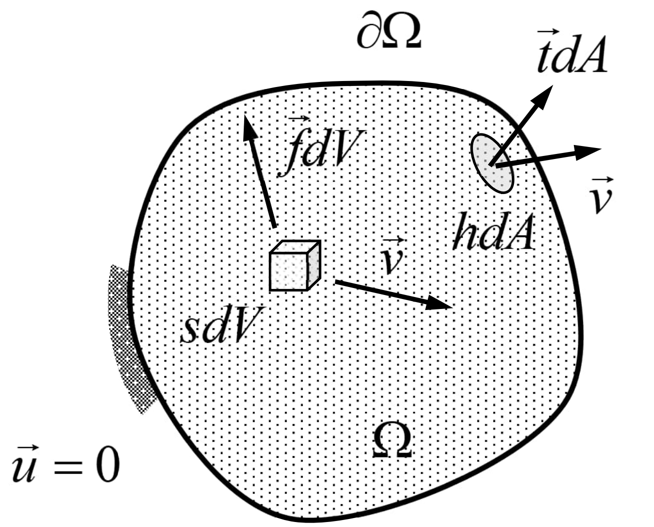
Temperature ϑ, heat Q, and internal energy U are concepts of continuum mechanics that do not have direct counterparts in particle mechanics (force and displacements have).

6.1 LINEAR THERMO-MECHANICS

Balance law	Local form in Ω	Local form on $\partial\Omega$
$\dfrac{Dm}{Dt} = 0$	$\rho^\circ = J\rho$	–
$\dfrac{D\vec{p}}{Dt} = \vec{F}$	$\rho^\circ \dfrac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \overleftrightarrow{\sigma} + \vec{f}$	$\vec{n} \cdot \overleftrightarrow{\sigma} = \vec{t}$
$\dfrac{D\vec{L}}{Dt} = \vec{M}$	$\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_{\mathrm{c}}$	–
$\dfrac{D(U+K)}{Dt} = P_W + P_Q$	$\rho^\circ \dfrac{\partial e}{\partial t} = \overleftrightarrow{\sigma} : \overleftrightarrow{d}_{\mathrm{c}} + s - \nabla \cdot \vec{q}$	$\vec{n} \cdot \vec{q} = h$

BOUNDARY VALUE PROBLEM

Given the initial stationary equilibrium temperature and displacement on Ω, the aim is to find new stationary equilibrium temperature and displacement, when external forces, heating etc. are changed in some manner.



Balance of momentun $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in Ω,

Balance of energy $-\nabla\cdot\vec{q}+s=0$ in Ω,

Displacement BC:s $\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t}$ or $\vec{u}=\vec{g}$ on $\partial\Omega$,

Temperature BC:s $\vec{n}\cdot\vec{q}=h$ or $\vartheta=\underline{\vartheta}$ on $\partial\Omega$.

Constitutive equations of the form $\vec{q}(\vartheta)$ (heat flux) and $\vec{\vec{\sigma}}(\vec{u},\vartheta)$ (stress) are needed for a closed equation system in terms of displacement and temperature.

GENERALIZED HOOKE'S LAW

The generalized Hooke's law, also considering the change of temperature $\Delta\vartheta = \vartheta - \vartheta^{\circ}$, is given by ($\vec{\sigma} = 0$ and $\Delta\vartheta = 0$ at the initial geometry)

Strain-stress: $$\begin{Bmatrix} \varepsilon_{xx} - \alpha\Delta\vartheta \\ \varepsilon_{yy} - \alpha\Delta\vartheta \\ \varepsilon_{zz} - \alpha\Delta\vartheta \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

Strain-displacement: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$$

Above, E is the Young's modulus, ν the Poisson's ratio, $G = E/(2+2\nu)$ the shear modulus, and α the thermal expansion coefficient. Strain and stress are symmetric.

FOURIER LAW OF HEAT CONDUCTION

When bodies at different temperatures are in contact, heat flows toward the cooler body until temperatures are the same. The Fourier law of heat conduction for an isotropic homogeneous material are (stress is assumed to vanish at the initial geometry) is given by

Heat-temperature: $$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix} \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \\ \partial\vartheta/\partial z \end{Bmatrix} = -k \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \\ \partial\vartheta/\partial z \end{Bmatrix}$$ Isotropic material

Thermal conductivity k($\left[\mathrm{N/(Ks)}\right]$ or $\left[\mathrm{W/(Km)}\right]$) depends on the material. The forms for the uni-axial and planar problems can be deduced from the generic form in the same manner as those for the stress-strain relationship.

EXAMPLE. Derive the stress-strain-temperature relationship of isotropic homogeneous material under (a) the $xy-$plane stress and (b) uni-axial stress conditions. Start with the generic strain-stress-temperature relationship.

Answer $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\sigma} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \alpha\Delta\vartheta \frac{E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$ and $\sigma_{xx} = E(\varepsilon_{xx} - \alpha\Delta\vartheta)$

- Under the plane stress assumption, only σ_{xx}, σ_{yy}, and σ_{xy} are non-zeros. The relationship for the in-plane normal stress resultants follows from the generic strain-temperature-stress relationship modified according to the kinetic assumption:

$$\begin{Bmatrix} \varepsilon_{xx} - \alpha\Delta\vartheta \\ \varepsilon_{yy} - \alpha\Delta\vartheta \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\sigma}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \alpha\Delta\vartheta\frac{E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}.$$

- Under the uni-axial stress assumption, only σ_{xx} is non-zero. The relationship follows directly from the generic strain-stress-temperature relationship. Inversion gives the stress-strain-temperature relationship for the uni-axial case

$$\varepsilon_{xx} - \alpha\Delta\vartheta = \frac{1}{E}\sigma_{xx} \quad \Leftrightarrow \quad \sigma_{xx} = E(\varepsilon_{xx} - \alpha\Delta\vartheta).$$

MATERIAL PARAMETERS

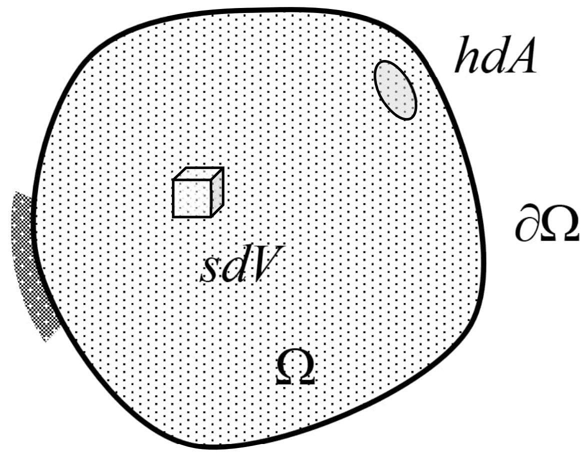
Material	ρ **[**$\mathrm{kg/m^3}$**]**	E [GPa]	ν **[** 1 **]**
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

MATERIAL PARAMETERS

Material	k **[**W / (Km)**]**	α **[**μm / mK**]**	c **[**J / kgK**]**
Steel	45…50	12…13	520
Aluminum	205…240	23…24	900
Copper	385…400	17	
Glass, ordinary	0.8…1	8…9	800
Granite	0.7…0.9		
Wood	0.1…0.2	30	1300
Rubber	0.2	0.1	
Concrete	1	12	850

VARIATIONAL REPRESENTATION

The variational form $\delta P = \delta P^{\text{int}} + \delta P^{\text{ext}} = 0 \quad \forall \delta\vartheta$ is the concise representation of the stationary heat conduction boundary value problem. In terms of density expressions $\delta p_{\Omega}^{\text{int}}$, $\delta p_{\Omega}^{\text{ext}}$, and $\delta p_{\partial\Omega}^{\text{ext}}$



Internal part: $\delta P^{\text{int}} = \int_{\Omega} \delta p_{\Omega}^{\text{int}} dV$,

External part: $\delta P^{\text{ext}} = \int_{\Omega} \delta p_{\Omega}^{\text{ext}} dV + \int_{\partial\Omega} \delta p_{\partial\Omega}^{\text{ext}} dA$.

The variational form lacks a clear physical interpretation although the meaning is clear from the mathematical viewpoint. The physical dimensions of δP $[\text{WK}]$ and δW $[\text{J}]$ differ, the former being power and the latter work.

- In derivation, the local form of energy balance is multiplied by $\delta\vartheta$, integrated over the domain followed by integration by parts in the heat flux term. Manipulations give the equivalent representations

$$-\nabla\cdot\vec{q}+s=0 \quad \text{in} \quad \Omega \quad \Leftrightarrow$$

$$\int_{\Omega}\delta\vartheta(-\nabla\cdot\vec{q}+s)dV=\int_{\Omega}(\nabla\delta\vartheta\cdot\vec{q}+\delta\vartheta s)dV-\int_{\partial\Omega}\delta\vartheta\vec{n}\cdot\vec{q}dA=0 \quad \forall\delta\vartheta.$$

- Assumption $\delta\vartheta=0$ (temperature specified) or $\vec{n}\cdot\vec{q}+h=0$ (heat flux specified) on $\partial\Omega$ gives the final form

$$\delta P=0 \quad \forall\delta\vartheta \quad \text{where} \quad \delta P=\int_{\Omega}\nabla\delta\vartheta\cdot\vec{q}dV+\int_{\Omega}\delta\vartheta s dV+\int_{\partial\Omega}\delta\vartheta h dA\,. \quad \Leftarrow$$

DENSITY EXPRESSIONS

The integrands of the variational form represent the model in the same manner as the virtual work densities in principle of virtual work:

Internal part: $$\delta p_{\Omega}^{\text{int}} = \begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \\ \partial\delta\vartheta/\partial z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \\ \partial\delta\vartheta/\partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \\ \partial\vartheta/\partial z \end{Bmatrix},$$ Isotropic material

External parts: $\delta p_{\Omega}^{\text{ext}} = \delta\vartheta s$ and $\delta p_{\partial\Omega}^{\text{ext}} = \delta\vartheta h$.

Thermal conductivity k $[\mathrm{W/(Km)}]$, power of heat per unit volume s $[\mathrm{W/m^3}]$, and power of heat per unit area h $[\mathrm{W/m^2}]$ may depend on position. For non-isotropic materials thermal conductivity is a (positive definite) matrix.

6.2 THERMO-MECHANICAL FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions $\delta W^{e} = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{cpl}}$ and $\delta P^{e} = \delta P^{\text{int}} + \delta P^{\text{ext}}$ in terms of nodal displacements/rotation components of the structural coordinate system and temperature.

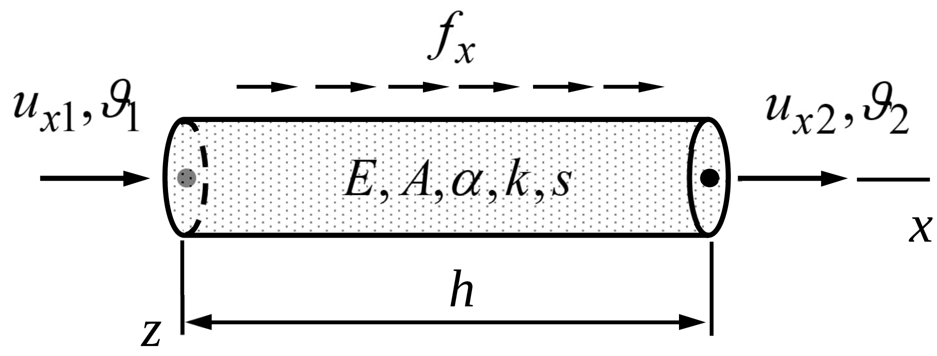
- Sum the element contributions to end up with the variational expression for the structure. Re-arrange to get $\delta W + \tau\delta P = -\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}(\mathbf{a},\mathbf{b}) - \tau\delta\mathbf{b}^{\mathrm{T}}\mathbf{R}(\mathbf{b})$ (τ is a dimensionally correct but otherwise arbitrary constant).

- Use the principle $\delta W + \tau\delta P = 0 \quad \forall \delta\mathbf{a}, \delta\mathbf{b}$ and the fundamental lemma of variation calculus to deduce $\mathbf{R}(\mathbf{a},\mathbf{b}) = 0$ and $\mathbf{R}(\mathbf{b}) = 0$. Solve the linear algebraic equations for the nodal displacements, rotations, and temperatures (due to the one-sided coupling of the stationary problem, solving the temperature first is always possible).

BAR MODE

Assuming that $v=0,\ w=0,\ \phi=0$ and a linear interpolation to the axial displacement $u(x)$ and temperature $\vartheta(x)$

$$\delta P^{\text{int}}=-\begin{Bmatrix}\delta\vartheta_1\\ \delta\vartheta_2\end{Bmatrix}^{\mathrm{T}}\frac{kA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\vartheta_1\\ \vartheta_2\end{Bmatrix},$$

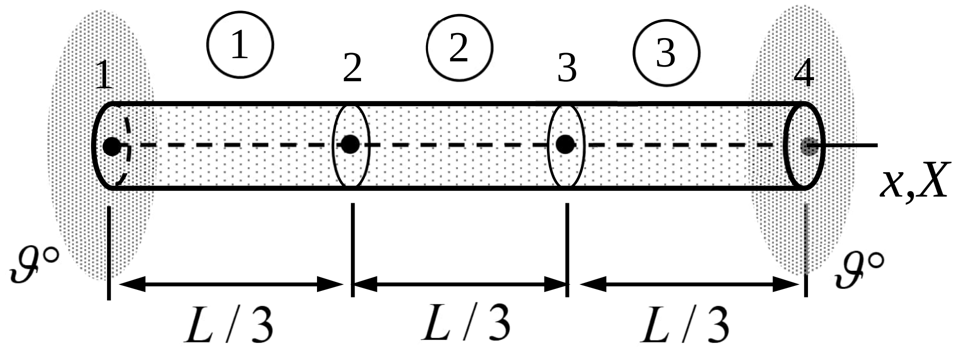


$$\delta W^{\text{cpl}}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_1\\ \Delta\vartheta_2\end{Bmatrix},$$

$$\delta P^{\text{ext}}=\begin{Bmatrix}\delta\vartheta_1\\ \delta\vartheta_2\end{Bmatrix}^{\mathrm{T}}\frac{Ash}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$

Heat flux through the end-planes is treated by point elements in the same manner as traction on the end-plates by point forces and moments.

EXAMPLE 6.1 The bar of the figure consists of three linear elements of identical lengths. Determine the stationary temperatures ϑ_2 at node 2 and ϑ_3 at node 3 when the end temperature is ϑ° and heat generation s per unit volume are constants. Take only the heat conduction along the bar axis into account. Problem parameters E, A, and k are constants.



Answer $\begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} = (\vartheta^\circ + \frac{1}{9}\frac{sL^2}{k})\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- Element contributions for the temperature distribution problem are (temperature is not affected by displacement)

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}, \quad \delta P^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{Ash}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- When the actual nodal values are substituted there, element contributions simplify to

$$\delta P^1 = -\begin{Bmatrix} 0 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \left(\frac{3kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_2 \end{Bmatrix} - \frac{AsL}{6} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right),$$

$$\delta P^2 = -\begin{Bmatrix} \delta\vartheta_2 \\ \delta\vartheta_3 \end{Bmatrix}^{\mathrm{T}} \left(\frac{3kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} - \frac{AsL}{6} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right),$$

$$\delta P^3 = -\begin{Bmatrix} \delta\vartheta_3 \\ 0 \end{Bmatrix}^{\mathrm{T}} \left(\frac{3kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_3 \\ \vartheta^{\circ} \end{Bmatrix} - \frac{AsL}{6}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right).$$

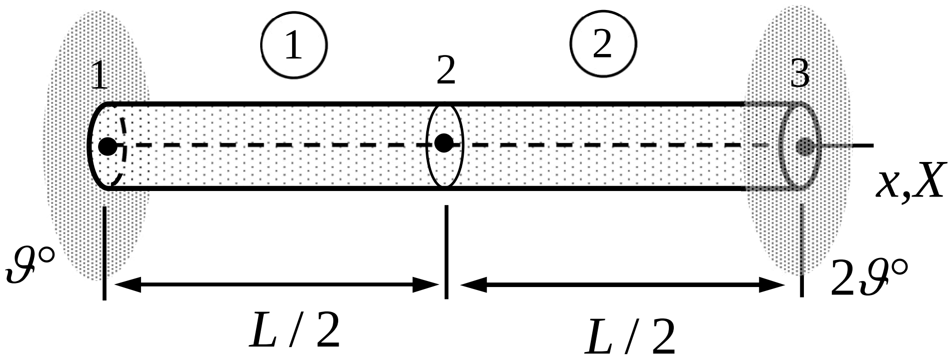
- Variational expression for a structure is the sum of the element contributions

$$\delta P = -\begin{Bmatrix} \delta\vartheta_2 \\ \delta\vartheta_3 \end{Bmatrix}^{\mathrm{T}} \left(\frac{3kA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} - \frac{3kA}{L}\begin{Bmatrix} \vartheta^{\circ} \\ \vartheta^{\circ} \end{Bmatrix} - \frac{AsL}{6}\begin{Bmatrix} 2 \\ 2 \end{Bmatrix}\right).$$

- Variational principle $\delta P = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply a linear equation system and thereby the solution

$$\frac{3kA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} - \frac{3kA}{L}\begin{Bmatrix} \vartheta^{\circ} \\ \vartheta^{\circ} \end{Bmatrix} - \frac{AsL}{6}\begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} = \left(\vartheta^{\circ} + \frac{1}{9}\frac{sL^2}{k}\right)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \leftarrow$$

EXAMPLE 6.2 The bar of the figure consists of two elements having the same material properties. Stress is zero, when the temperature in the wall and bar is ϑ°. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the right end is increased to $2\vartheta^\circ$. Take only the heat conduction along the bar axis into account. Use two linear elements. Problem parameters E, A, k, and α are constants.



Answer $u_{X2} = -\dfrac{1}{8} L\alpha\vartheta^\circ$ and $\vartheta_2 = \dfrac{3}{2}\vartheta^\circ$

- Element contributions for the thermo-mechanical problem needed in this case are (no heat production, nor external distributed forces, and $\Delta\vartheta = \vartheta - \vartheta^{\circ}$).

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \vartheta_2 - \vartheta^{\circ} \end{Bmatrix},$$

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\text{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}.$$

- As the nodal values for bar 1 are $u_{x1} = 0$, $u_{x2} = u_{X2}$, $\Delta\vartheta_1 = 0$, and $\Delta\vartheta_2 = \vartheta_2 - \vartheta^{\circ}$, the element contributions $\delta W^{\text{int}} + \delta W^{\text{cpl}}$ and δP^{int} simplify to

$$\delta W^{1} = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\text{T}} \frac{2EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\text{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \vartheta_2 - \vartheta^{\circ} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta W^1 = -\delta u_{X2} \frac{2EA}{L} u_{X2} + \delta u_{X2} \frac{\alpha EA}{L} (\vartheta_2 - \vartheta^\circ),$$

$$\delta P^1 = -\begin{Bmatrix} 0 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{2kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_2 \end{Bmatrix} = -\delta\vartheta_2 \frac{2kA}{L} (\vartheta_2 - \vartheta^\circ).$$

- As the nodal values for bar 2 are $u_{x3} = 0$, $u_{x2} = u_{X2}$, $\Delta\vartheta_3 = 2\vartheta^\circ - \vartheta^\circ = \vartheta^\circ$, and $\Delta\vartheta_2 = \vartheta_2 - \vartheta^\circ$, the element contributions $\delta W^{\mathrm{int}} + \delta W^{\mathrm{cpl}}$ and δP^{int} simplify to

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{2EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_2 - \vartheta^\circ \\ \vartheta^\circ \end{Bmatrix} \Leftrightarrow$$

$$\delta W^2 = -\delta u_{X2} \frac{2EA}{L} u_{X2} - \delta u_{X2} \frac{\alpha EA}{2} \vartheta_2,$$

$$\delta P^2 = -\begin{Bmatrix} \delta\vartheta_2 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{2kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_2 \\ 2\vartheta^\circ \end{Bmatrix} = -\delta\vartheta_2 \frac{2kA}{L} (\vartheta_2 - 2\vartheta^\circ).$$

- Variational expressions for the mechanical and thermal parts are sums of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{X2}(\frac{4EA}{L}u_{X2} + \frac{\alpha EA}{2}\vartheta^{\circ}),$$

$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_2 \frac{2kA}{L}(2\vartheta_2 - 3\vartheta^{\circ}).$$

- Variational principle $\delta W + \tau\delta P = 0 \quad \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the equations

$$\frac{4EA}{L}u_{X2} + \frac{\alpha EA}{2}\vartheta^{\circ} = 0 \text{ and } \frac{2kA}{L}(2\vartheta_2 - 3\vartheta^{\circ}) = 0 \quad \Leftrightarrow$$

$$\vartheta_2 = \frac{3}{2}\vartheta^{\circ} \text{ and } u_{X2} = -\frac{\alpha L}{8}\vartheta^{\circ}. \quad \leftarrow$$

- In Mathematica notation, the problem description is given by

	model	properties	geometry
1	BAR	$\{\{E, \alpha, k\}, \{A\}, \{\{0, \vartheta 0\}\}\}$	Line[{1, 2}]
2	BAR	$\{\{E, \alpha, k\}, \{A\}, \{\{0, \vartheta 0\}\}\}$	Line[{2, 3}]

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\Theta_X, \Theta_Y, \Theta_Z\}$	ϑ
1	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\vartheta 0$
2	$\{\frac{L}{2}, 0, 0\}$	$\{uX[2], 0, 0\}$	$\{0, 0, 0\}$	$\vartheta[2]$
3	$\{L, 0, 0\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$2\,\vartheta 0$

$$\left\{uX[2] \to -\frac{1}{8} L\, \alpha\, \vartheta 0,\ \vartheta[2] \to \frac{3\,\vartheta 0}{2}\right\}$$

6.3 ELEMENT CONTRIBUTIONS

Variational expressions for the elements combine the density expressions of a model and approximations depending on the element shape and type. To derive the expression for an element:

- Start with the densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{ext}}$, $\delta w_{\Omega}^{\text{cpl}}$, $\delta p_{\Omega}^{\text{int}}$, and $\delta p_{\Omega}^{\text{ext}}$ of the model. If not given in the formulae collection, derive the expressions starting from the 3D versions.

- Represent the unknown functions by interpolation of the nodal displacements, rotations, and temperatures. Substitute the approximations into the density expressions.

- Integrate the densities over the domain occupied by the element to end up with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{cpl}}$ and $\delta P = \delta P^{\text{int}} + \delta P^{\text{ext}}$

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacements and rotations in terms of shape functions. In thermo-mechanical analysis, temperature is represented in the same manner by using nodal temperatures.

Approximation $u = \mathbf{N}^{\mathrm{T}}\mathbf{a},\ v = \mathbf{N}^{\mathrm{T}}\mathbf{a},\ \ldots.,\ \vartheta = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

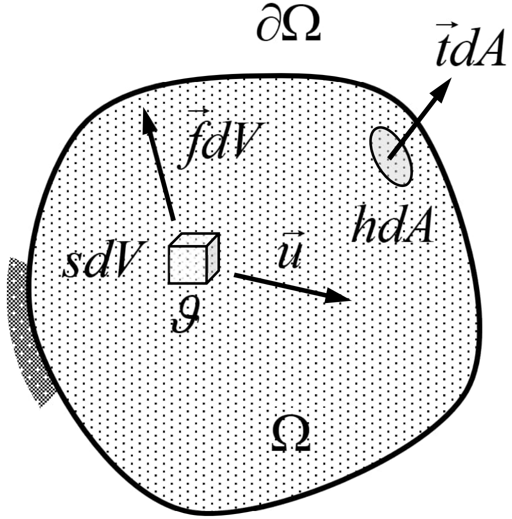
Shape functions $\mathbf{N} = \{N_1(x,y,z) \quad N_2(x,y,z) \quad \ldots \quad N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \ldots \quad \mathrm{a}_n\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z, \vartheta\}$ may be just displacement or rotation components or a mixture of them (as with the Bernoulli beam model). Nodal parameters may also represent temperature.

SOLID MODEL

The model does not contain any kinetic or kinematic assumptions. Virtual work densities of the internal and external distributed forces $\delta w_{\Omega}^{\text{int}}$ and $\delta w_{\Omega}^{\text{ext}}$ are the same as in linear displacement analysis. The additional terms are

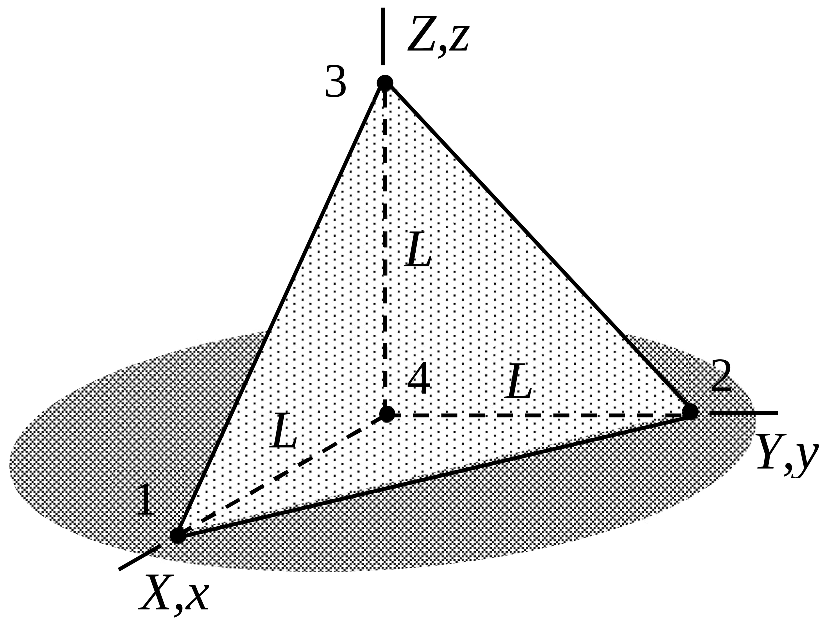


$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta w / \partial z \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha\Delta\vartheta}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix},$$

$$\delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \\ \partial\vartheta / \partial z \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s.$$

The solution domain can be represented, e.g., by tetrahedron elements with linear interpolation of $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ and $\vartheta(x,y,z)$.

EXAMPLE 6.3 Consider a tetrahedron of edge length L on a horizontal floor. Determine displacement u_{Z3} when temperature is increased by constant $\Delta\vartheta$ and before that stress vanishes. Assume that $u_{X3} = u_{Y3} = 0$ and that the bottom surface is fixed. Stress vanishes at the initial geometry when $u_{Z3} = 0$. Material parameters E, $\nu = 0$, and α are constants.



Answer: $u_{Z3} = L\alpha\Delta\vartheta$

- Only the shape function $N_3 = z / L$ of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ v = 0, \text{ and } w = \frac{z}{L} u_{Z3},\ \text{ giving } \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0, \text{ and } \frac{\partial w}{\partial z} = \frac{1}{L} u_{Z3}.$$

- As temperature is known, it is enough to consider the displacement problem. With the approximation, the internal and coupling densities simplify to ($\nu = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3} / L \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z3} / L \end{Bmatrix} = -\frac{E}{L^2} u_{Z3} \delta u_{Z3},$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3} / L \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha\Delta\vartheta}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{\delta u_{Z3}}{L} E\alpha\Delta\vartheta.$$

- Virtual work expressions are integrals of the densities over the volume. Here, the densities are constants, and it is enough to multiply by the volume $L^3 / 6$

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dV = -\delta u_{Z3} \frac{1}{6} E L u_{Z3},$$

$$\delta W^{\text{cpl}} = \int_{\Omega} \delta w_{\Omega}^{\text{cpl}} dV = \delta u_{Z3} \frac{1}{6} L^2 E \alpha \Delta \vartheta.$$

- Variational principle (here principle of virtual work) $\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = 0$ implies that

$$-\frac{1}{6} E L u_{Z3} + \frac{1}{6} L^2 E \alpha \Delta \vartheta = 0 \quad \Leftrightarrow \quad u_{Z3} = L \alpha \Delta \vartheta.$$

PLATE MODEL

Virtual work densities combine the plane-stress and plate bending modes. Assuming that the material coordinate system is placed at the mid-plane, and material properties do not depend on the transverse coordinate,

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \end{Bmatrix}^{\text{T}} \int \Delta\vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} \partial^2\delta w / \partial x^2 \\ \partial^2\delta w / \partial y^2 \end{Bmatrix}^{\text{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\text{T}} k \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \\ \partial\vartheta / \partial z \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s \quad \text{and} \quad \delta p_{\partial\Omega}^{\text{ext}} = \delta\vartheta h.$$

Approximation to the transverse displacement depends only on the planar coordinates but temperature and its approximation may depend on all the coordinates.

- The constitutive equations of a linearly elastic isotropic material and kinetic assumption $\sigma_{zz} = 0$ give the non-zero stress components

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\sigma} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \alpha \Delta \vartheta \frac{E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \text{with} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} - z \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}.$$

- The generic expression of $\delta w_{\Omega}^{\text{int}}$ simplifies to a sum of thin slab, bending and interaction parts. Assuming that material properties do not depend on z, and that the origin of the material coordinate system is placed at the mid-plane, virtual work density of internal forces consists of the internal parts of the plate thin-slab and bending modes $\delta w_{\Omega}^{\text{int}}$ and the coupling parts for the thin-slab and bending modes (the integral is over the thickness)

$$\delta w_{\Omega}^{\mathrm{cpl}} = \begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \end{Bmatrix}^{\mathrm{T}} \int \Delta \vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

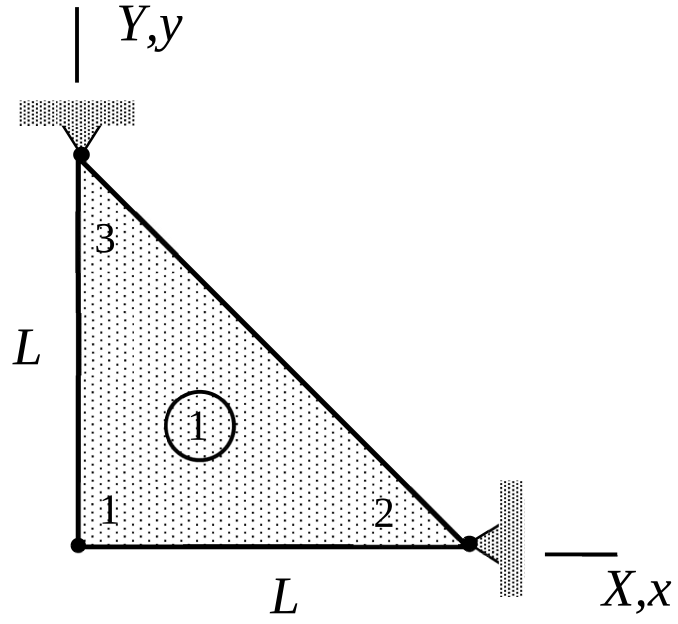
$$\delta w_{\Omega}^{\mathrm{cpl}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z \Delta \vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$

- As temperature is not assumed to be constant in the thickness direction, variational expression for the temperature calculation is based on the generic expressions. Therefore, also the approximation, e.g., of the type

$$\vartheta(x, y, z) = \mathbf{N}^{\mathrm{T}}(x, y)\mathbf{a}(z) \quad \text{where} \quad \mathbf{a}(z) = \mathbf{a}_0 + \mathbf{a}_z z$$

is used for the actual domain of the plate.

EXAMPLE 6.4 Consider the triangular thin slab shown. Determine displacements u_{X1} and u_{Y1}, when temperature is increased by constant $\Delta\vartheta$ and before that stress vanishes. Use a linear approximation and assume plane stress conditions. Thickness of the slab is t and material parameters E, ν, and α are constants.



Answer $\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{1+\nu}{2} L\alpha\Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- The non-zero displacement components are $u_{x1} = u_{X1}$ and $u_{y1} = u_{Y1}$. The linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ can be deduced from the figure. Therefore, approximations are

$$u = N_1 u_{x1} = \frac{1}{L}(L - x - y)u_{X1} \ \text{ and } \ v = N_1 u_{y1} = \frac{1}{L}(L - x - y)u_{Y1} \ \Rightarrow$$

$$\frac{\partial u}{\partial x} = -\frac{u_{X1}}{L}, \ \frac{\partial u}{\partial y} = -\frac{u_{X1}}{L}, \ \frac{\partial v}{\partial x} = -\frac{u_{Y1}}{L} \ \text{ and } \ \frac{\partial v}{\partial y} = -\frac{u_{Y1}}{L}.$$

- Densities of internal and coupling terms simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\delta u_{X1} \\ -\delta u_{Y1} \\ -\delta u_{X1} - \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -u_{X1} \\ -u_{Y1} \\ -u_{X1} - u_{Y1} \end{Bmatrix} \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} + \frac{Et}{2(1+\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \frac{1}{L^2},$$

$$\delta w_{\Omega}^{\text{cpl}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L} \frac{E\alpha t}{1-\nu} \Delta \vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- Integration over the element gives (densities are constants)

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{Et}{2(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} + \frac{Et}{4(1+\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta W^{\text{cpl}} = \int_{\Omega} \delta w_{\Omega}^{\text{cpl}} dA = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{L}{2} \frac{E\alpha t}{1-\nu} \Delta \vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

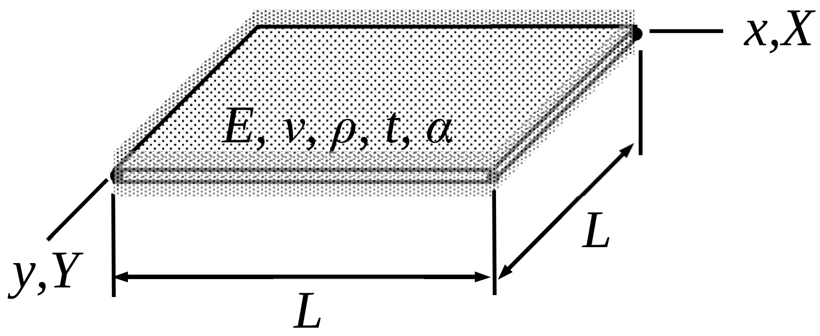
- Variation principle $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \ \forall \delta \mathbf{a}$ and fundamental lemma of variation calculus imply the equilibrium equations

$$\left(\frac{Et}{2(1-\nu^2)}\begin{bmatrix}1 & \nu \\ \nu & 1\end{bmatrix}+\frac{Et}{4(1+\nu)}\begin{bmatrix}1 & 1 \\ 1 & 1\end{bmatrix}\right)\begin{Bmatrix}u_{X1} \\ u_{Y1}\end{Bmatrix}+\frac{L}{2}\frac{E\alpha t}{1-\nu}\Delta\vartheta\begin{Bmatrix}1 \\ 1\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{X1} \\ u_{Y1}\end{Bmatrix}=-\begin{bmatrix}1/(1-\nu)+1/2 & \nu/(1-\nu)+1/2 \\ \nu/(1-\nu)+1/2 & 1/(1-\nu)+1/2\end{bmatrix}^{-1}\frac{1+\nu}{1-\nu}L\alpha\Delta\vartheta\begin{Bmatrix}1 \\ 1\end{Bmatrix} \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{X1} \\ u_{Y1}\end{Bmatrix}=-\frac{1+\nu}{2}L\alpha\Delta\vartheta\begin{Bmatrix}1 \\ 1\end{Bmatrix}. \quad \leftarrow$$

EXAMPLE 6.5 Simply supported plate of the figure is assembled at constant temperature $3\vartheta°$. Find the transverse displacement when the upper side temperature is $4\vartheta°$ and that of the lower side $2\vartheta°$. Assume that temperature in plate is linear in z. Use the polynomial approximation $w(x,y) = \mathrm{a}(xy/L^2)(1-x/L)(1-y/L)$. Problem parameters E, ν, ρ, α and t are constants.



Answer $w(x,y) = -\dfrac{30}{11}\alpha\vartheta°(1+\nu)\dfrac{L^2}{t}\dfrac{xy}{L^2}(1-\dfrac{x}{L})(1-\dfrac{y}{L})$

- Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the bending mode virtual work densities of the internal and coupling parts are given by

$$\delta w_\Omega^{\rm int} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\rm T} D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} \text{ where } D = \frac{t^3}{12}\frac{E}{1-\nu^2},$$

$$\delta w_\Omega^{\rm cpl} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\rm T} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- Approximation to the transverse displacement and its derivatives are

$$w(x,y) = \mathrm{a}\frac{xy}{L^2}(1-\frac{x}{L})(1-\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = -2\mathrm{a}\frac{y}{L^3}(1-\frac{y}{L}),\ \ \frac{\partial^2 w}{\partial y^2} = -2\mathrm{a}\frac{x}{L^3}(1-\frac{x}{L}),\ \ \frac{\partial^2 w}{\partial x \partial y} = \mathrm{a}\frac{1}{L^2}(1-2\frac{x}{L})(1-2\frac{y}{L}).$$

- Temperature difference and its weighted integral over the thickness (integral of the coupling term)

$$\Delta\vartheta = \vartheta(z) - 3\vartheta^\circ = (\frac{1}{2}+\frac{z}{t})2\vartheta^\circ + (\frac{1}{2}-\frac{z}{t})4\vartheta^\circ - 3\vartheta^\circ = -\frac{z}{t}2\vartheta^\circ \quad \Rightarrow$$

$$\int z\Delta\vartheta dz = -\int_{-t/2}^{t/2} z\frac{z}{t}2\vartheta^\circ dz = -\frac{1}{6}\vartheta^\circ t^2 \ .$$

- When the approximation is substituted there, virtual work expressions of the internal and coupling terms simplify to

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta \mathrm{a} \frac{22}{45} \frac{1}{L^2} \frac{t^3}{12} \frac{E}{1-\nu^2} \mathrm{a} \,,$$

$$\delta W^{\text{cpl}} = \int_0^L \int_0^L \delta w_\Omega^{\text{cpl}} dxdy = -\delta \mathrm{a} \frac{1}{9} \frac{\alpha E}{1-\nu} \vartheta^\circ t^2 .$$

- Virtual work expression is the sum of the internal and coupling parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta \mathrm{a} \left(\frac{22}{45} \frac{1}{L^2} \frac{t^3}{12} \frac{E}{1-\nu^2} \mathrm{a} + \frac{1}{9} \frac{\alpha E}{1-\nu} \vartheta^\circ t^2 \right).$$

- Principle of virtual work $\delta W = 0 \,\forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\mathrm{a} = -\frac{30}{11} \alpha \vartheta^\circ (1+\nu) \frac{L^2}{t} \quad \Rightarrow \quad w(x,y) = -\frac{30}{11} \alpha \vartheta^\circ (1+\nu) \frac{L^2}{t} \frac{xy}{L^2} (1-\frac{x}{L})(1-\frac{y}{L}) \;.$$

BEAM MODEL

Virtual work densities combine the bar, bending, and torsion modes. Assuming that material properties are constants, and the material coordinate system is placed so that the first and the cross moments of the cross section vanish

$$\delta w_{\Omega}^{\mathrm{cpl}} = E\alpha(\begin{Bmatrix} d\delta u / dx \\ d^2\delta v / dx^2 \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} \int \Delta\vartheta \begin{Bmatrix} 1 \\ -y \\ -z \end{Bmatrix} dA, \quad \delta p_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \\ \partial\vartheta / \partial z \end{Bmatrix}, \quad \text{and}$$

$$\delta p_{\Omega}^{\mathrm{ext}} = \delta\vartheta s \quad \text{and} \quad \delta p_{\partial\Omega}^{\mathrm{ext}} = \delta\vartheta h.$$

Approximation to the transverse displacement depends only on the axial coordinate but temperature and its approximation may depend on all the coordinates in the expressions.

- The displacement components of the Bernoulli beam model are $u_x = u-(dw/dx)z-(dv/dx)y$, $u_y = v-\phi z$ and $u_z = w+\phi y$. With the kinetic assumption $\sigma_{zz} = \sigma_{yy} = 0$, stress and strain components take the forms

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} - E\alpha\Delta\vartheta \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \text{ where } \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{du}{dx} - \dfrac{d^2w}{dx^2}z - \dfrac{d^2v}{dx^2}y \\ -z\dfrac{d\phi}{dx} \\ y\dfrac{d\phi}{dx} \end{Bmatrix}.$$

- Assuming that material properties are constants, and the material coordinate system is placed so that the first and the cross moments of the cross section vanish, the virtual work density of the coupling term simplifies to (after integration over the cross section)

$$\delta w_{\Omega}^{\mathrm{cpl}} = E\alpha\left(\frac{d\delta u}{dx}\int \Delta\vartheta dA - \frac{d^2\delta w}{dx^2}\int z\Delta\vartheta dA - \frac{d^2\delta v}{dx^2}\int y\Delta\vartheta dA\right). \quad \leftarrow$$

- As temperature is not assumed to be constant in the thickness direction, variational expression for the temperature calculation is based on the generic expressions. Accordingly, the approximation depends on all the coordinates. Approximation of the type

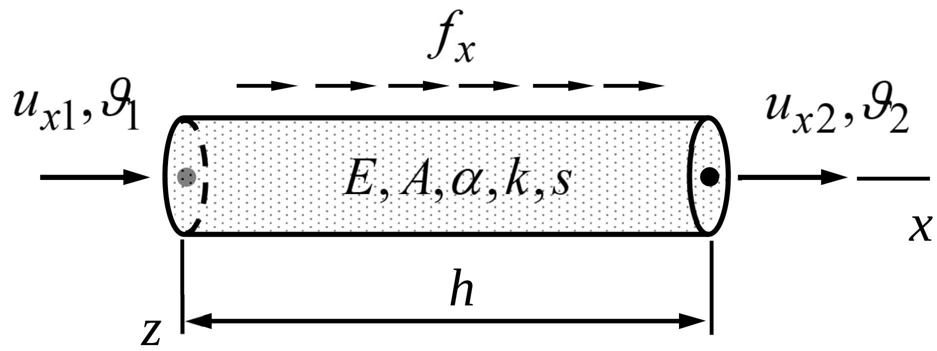
$$\vartheta(x,y,z) = \mathbf{N}^{\mathrm{T}}(x)\mathbf{a}(y,z) \quad \text{where} \quad \mathbf{a}(y,z) = \mathbf{a}_0 + \mathbf{a}_y y + \mathbf{a}_z z$$

is one of the possibilities.

BAR MODE

Assuming that $v = 0,\ w = 0,\ \phi = 0$ and a linear interpolation to the axial displacement $u(x)$ and temperature $\vartheta(x)$

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix},$$



$$\delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix},$$

$$\delta P^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{Ash}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Heat flux through the end-planes is treated by point elements in the same manner as traction on the end-plates by point forces and moments.

- Bar model assumes that $v(x)=w(x)=0$ or that coupling between the bar and bending modes vanish. After integration over the cross section, the generic expressions for the 3D case simplify to

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} \ , \ \delta w_\Omega^{\text{ext}} = \delta u f_x, \ \ \delta w_\Omega^{\text{cpl}} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta,$$

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx}, \ \ \delta p_\Omega^{\text{ext}} = \delta\vartheta s,$$

in which cross-sectional area A, Young's modulus E, external force per unit length f_x, thermal conductivity k, coefficient of thermal expansion α, and heat production rate per unit length s may depend on x.

- Linear interpolants to the axial displacement and temperature are

$$u=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix},\ \vartheta=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}\vartheta_1\\\vartheta_2\end{Bmatrix}, \text{ and } \Delta\vartheta=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}\Delta\vartheta_1\\\Delta\vartheta_2\end{Bmatrix}.$$

- After substituting the approximations into the densities and integration over the domain occupied by the element with the assumedly constant material properties

$$\delta W^{\text{cpl}}=\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_1\\\Delta\vartheta_2\end{Bmatrix},$$

$$\delta P^{\text{int}}=-\begin{Bmatrix}\delta\vartheta_1\\\delta\vartheta_2\end{Bmatrix}^{\mathrm{T}}\frac{kA}{h}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}\vartheta_1\\\vartheta_2\end{Bmatrix},\ \delta P^{\text{ext}}=\begin{Bmatrix}\delta\vartheta_1\\\delta\vartheta_2\end{Bmatrix}^{\mathrm{T}}\frac{sAh}{2}\begin{Bmatrix}1\\1\end{Bmatrix}. \quad \Leftarrow$$

Assuming a cubic interpolation to $w(x)$ and $v(x)$ and linear interpolation to the "coefficients" of the representation $\Delta\vartheta(x,z) = \Delta\vartheta_0(x) + \Delta\vartheta_y(x)y + \Delta\vartheta_z(x)z$, the coupling term

$$\delta W^{\mathrm{cpl}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}\alpha}{h^2} \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} \\ -1 & 0 \\ \frac{1}{h} & -\frac{1}{h} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta_{y1} \\ \Delta\vartheta_{y2} \end{Bmatrix} - \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}\alpha}{h^2} \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} \\ 1 & 0 \\ \frac{1}{h} & -\frac{1}{h} \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta_{z1} \\ \Delta\vartheta_{z2} \end{Bmatrix}$$

Under the assumptions used, the displacement-temperature coupling of the bar and the bending modes can be treated by adding a coupling term for each mode.

- Cubic interpolants to the transverse displacements and the "Taylor series" type linear approximation to the temperature difference are

$$v=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ h(1-\xi)^2\xi\\ (3-2\xi)\xi^2\\ h\xi^2(\xi-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\\ u_{y2}\\ \theta_{z2}\end{Bmatrix},\quad w=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ -h(1-\xi)^2\xi\\ (3-2\xi)\xi^2\\ -h\xi^2(\xi-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix},$$

$$\Delta\vartheta=\begin{Bmatrix}1-\xi\\ \xi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\Delta\vartheta_1\\ \Delta\vartheta_2\end{Bmatrix}+y\begin{Bmatrix}1-\xi\\ \xi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\Delta\vartheta_{y1}\\ \Delta\vartheta_{y2}\end{Bmatrix}+z\begin{Bmatrix}1-\xi\\ \xi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\Delta\vartheta_{z1}\\ \Delta\vartheta_{z2}\end{Bmatrix}\ \text{where } \xi=\frac{x}{h}.$$

- When the approximation is substituted there, integration of the density over the cross sections gives the coupling expression (notice that the first term of the temperature approximation contributes to the bar mode only).