ELEC-C8201 - Control and automation

Frequency Response Methods

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In the previous lecture...

You:

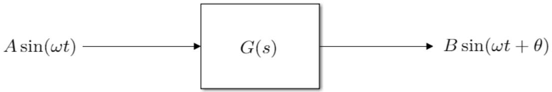
- Understood the concept of the root locus and its role in control system design
- Knew how to obtain a root locus plot by sketching or using MATLAB
- Got familiar with the PID controller as a key element of many feedback systems

Learning outcomes...

...the student will:

- Understand the powerful concept of frequency response and its role in control system design
- Know how to sketch a Bode plot and also how to obtain a computer-generated Bode plot
- Become familiar with log magnitude and phase diagrams

Response to a Sinusoidal Input Signal



$$u(t) = A\sin(\omega t) \quad , \quad y(t) = B\sin(\omega t + \theta)$$

where

$$B = A|G(j\omega)| \quad , \quad \theta = \angle G(j\omega)$$

- The response of a linear constant coefficient system to a sinusoidal input signal is an output sinusoidal signal at the same frequency as the input.
- However, the magnitude and phase of the output signal differ from those of the input sinusoidal signal, and the amount of difference is a function of the input frequency.

Frequency Response Plots

The transfer function of a system $G(s)$ can be described in the frequency domain by:

1)

$$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega), \tag{1}$$

where

$$R(\omega) = Re[G(j\omega)] \quad \text{and} \quad X(\omega) = Im[G(j\omega)].$$

2)

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} = |G(j\omega)|\angle\phi(\omega), \tag{2}$$

where

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)} \quad \text{and} \quad |G(j\omega)|^2 = [R(\omega)]^2 + [X(\omega)]^2$$

Bode Plots

The limitations of polar plots are readily apparent:

- The addition of poles or zeros to an existing system requires the recalculation of the frequency response
- Calculating the frequency response in this manner is tedious and does not indicate the effect of the individual poles or zeros.

- **Logarithmic plots**, often called **Bode plots**, simplifies the determination of the graphical portrayal of the frequency response!

Bode Plots - Generalized transfer function

$$G(s) = \frac{K \prod_{i=1}^{Q} (1 + s\tau_i)}{(s)^N \prod_{m=1}^{M} (1 + s\tau_m) \prod_{k=1}^{R} \left[\left(1 + (2\zeta_k/\omega_{nk})\, s + (s/\omega_{nk})^2 \right) \right]} \tag{3}$$

After substituting s by $j\omega$:

$$G(j\omega) = \frac{K \prod_{i=1}^{Q} (1 + j\omega\tau_i)}{(j\omega)^N \prod_{m=1}^{M} (1 + j\omega\tau_m) \prod_{k=1}^{R} \left[\left(1 + (2\zeta_k/\omega_{nk})\, j\omega + (j\omega/\omega_{nk})^2 \right) \right]} \tag{4}$$

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Standard form of

$$G(s)=\frac{20(s+5)}{s^2(s+3)(s+10)}$$

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Standard form of

$$G(s) = \frac{20(s+5)}{s^2(s+3)(s+10)}$$

is

$$G(s) = \frac{\frac{10}{3}(\frac{1}{5}s + 1)}{s^2(\frac{1}{3}s + 1)(\frac{1}{10}s + 1)}$$

Bode Plots - Generalized transfer function

- The transfer function in the frequency domain is

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} \tag{5}$$

- The magnitude is normally expressed in terms of the logarithm to the base 10, so we use

$$\text{Logarithmic gain } = 20\log_{10}|G(j\omega)|dB \tag{6}$$

- what is the advantage of the logarithmic plot??

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- what is the advantage of the logarithmic plot??

  the conversion of multiplicative factors, into additive factors.

Bode Plots - Logarithmic magnitude & Angle plot

$$G(j\omega) = \frac{K\prod_{i=1}^{Q}(1+j\omega\tau_i)}{(j\omega)^N\prod_{m=1}^{M}(1+j\omega\tau_m)\prod_{k=1}^{R}\left[\left(1+(2\zeta_k/\omega_{nk})\,j\omega+(j\omega/\omega_{nk})^2\right)\right]}$$

The logarithmic magnitude of $G(j\omega)$ is

$$\begin{aligned} 20\log|G(j\omega)| =& 20\log K_b + 20\sum_{i=1}^{Q}\log|1+j\omega\tau_i| \\ & - 20\log\left|(j\omega)^N\right| - 20\sum_{m=1}^{M}\log|1+j\omega\tau_m| \\ & - 20\sum_{k=1}^{R}\log\left|1+\frac{2\zeta_k}{\omega_{nk}}j\omega+\left(\frac{j\omega}{\omega_{nk}}\right)^2\right| \end{aligned} \quad (7)$$

Furthermore, the separate phase angle plot is obtained as :

$$\phi(\omega) = +\sum_{i=1}^{Q}\tan^{-1}(\omega\tau_i) - N(90^\circ) - \sum_{m=1}^{M}\tan^{-1}(\omega\tau_m) - \sum_{k=1}^{R}\tan^{-1}\frac{2\zeta_k\omega_{nk}\omega}{\omega_{nk}^2-\omega^2} \quad (8)$$

Bode Plots Using Separate Factors of Transfer Functions

- Constant gain

$$K$$

- Poles (or zeros) at the origin

$$(j\omega)^{\pm 1}$$

- Poles (or zeros) on the real axis

$$(1 + j\omega\tau)^{\pm 1}$$

- Complex conjugate poles (or zeros)

$$(1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2)^{\pm 1}$$

Constant Gain K

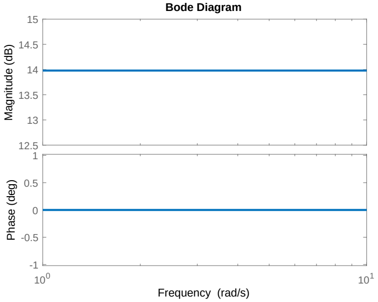
- The logarithmic gain is:

$$20\log_{10} K = \text{constant in dB}$$

- The phase angle is:

$$\phi(\omega) = 0$$

- Example: For $G(s) = 5$:



Poles at the origin $1/(j\omega)$

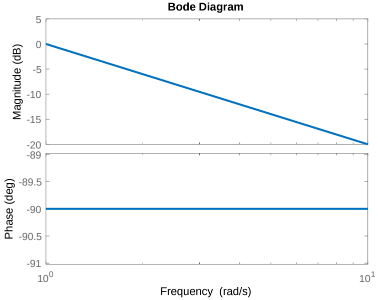
- The logarithmic gain and the phase angle are

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega\ dB \quad , \quad \phi(\omega) = -90^\circ$$

- The slope of the magnitude curve is -20 dB/decade for a pole
- For a multiple pole at the origin, we have :

$$20\log\left|\frac{1}{(j\omega)^N}\right| = -20N\log\omega\ dB \quad , \quad \phi(\omega) = -90^\circ N$$

- Example: For $G(s) = \frac{1}{s}$:



Zeros at the origin $(j\omega)$

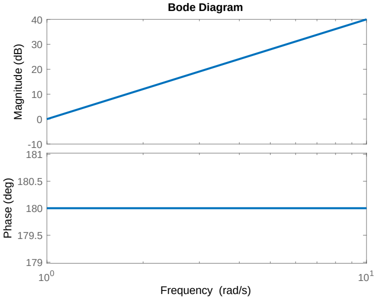
- The logarithmic gain and the phase angle are

$$20\log|j\omega| = +20\log\omega \ dB \quad , \quad \phi(\omega) = +90^\circ$$

- The slope of the magnitude curve is $+20$ dB/decade for a zero
- For a multiple zero at the origin, we have :

$$20\log\left|(j\omega)^N\right| = +20N\log\omega \ dB \quad , \quad \phi(\omega) = +90^\circ N$$

- Example: For $G(s) = s^2$:

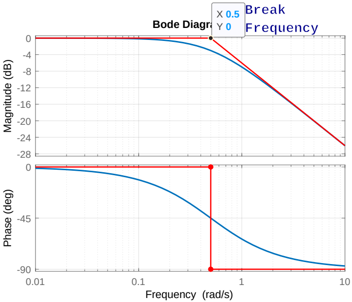


Poles on the real axis $1/(1+j\omega\tau)$

- The logarithmic gain and the phase angle for $1/(1+j\omega\tau)$ are

$$20\log\left|\frac{1}{1+j\omega\tau}\right| = -10\log(1+\omega^2\tau^2)\ dB \quad , \quad \phi(\omega) = -tan^{-1}(\omega\tau)$$

- **Break Frequency** or **Corner Frequency**: $\omega = 1/\tau$
- The asymptotic curve for $\omega \ll 1/\tau$ is $20\log 1 = 0$
- The asymptotic curve for $\omega \gg 1/\tau$ is $-20\log(\omega\tau)$, which has a slope of -20 dB/decade
- Example: For $G(s) = \frac{1}{1+2s}$: ($\omega = 1/2$ is the break frequency)

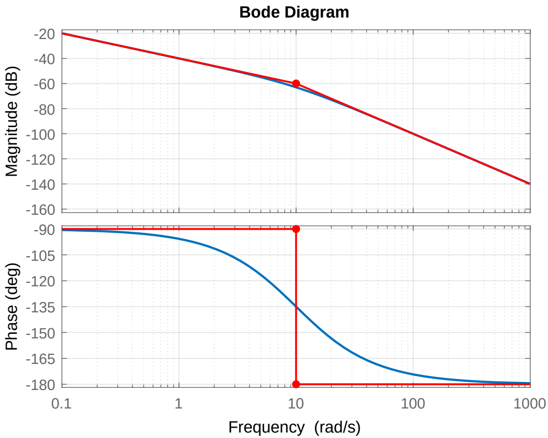


Example 1

- Bode plot of $G(s) = \frac{1}{10s(s+10)}$

Example 1

- Bode plot of $G(s) = \frac{1}{10s(s+10)}$
- Break frequencies: $\omega = 10$ rad/s



Example 2

- Bode plot of $G(s) = \frac{10}{(s+1)(10s+1)}$

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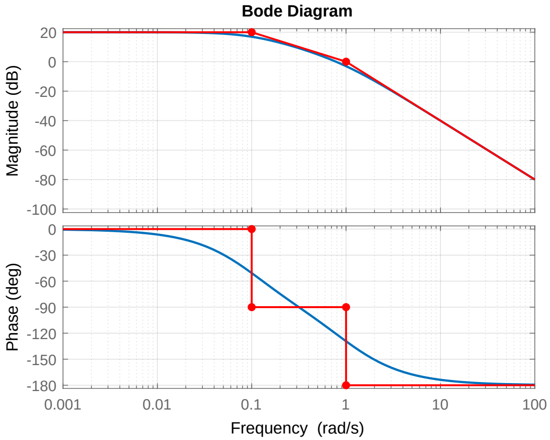
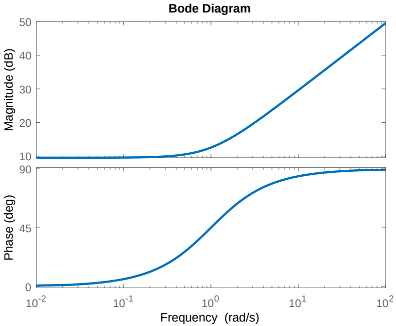
- Bode plot of $G(s) = \frac{10}{(s+1)(10s+1)}$
- Break frequencies: $\omega = 0.1$ and $\omega = 1$



Zeros on the real axis $(1 + j\omega\tau)$

- The logarithmic gain and the phase angle for $(1 + j\omega)$ are
$$20\log|1 + j\omega\tau| = +10\log(1 + \omega^2\tau^2)\ dB \quad , \quad \phi(\omega) = +tan^{-1}(\omega\tau)$$
- **Break Frequency** or **Corner Frequency**: $\omega = 1/\tau$
- The asymptotic curve for $\omega \ll 1/\tau$ is $20\log 1 = 0$
- The asymptotic curve for $\omega \gg 1/\tau$ is $20\log(\omega\tau)$, which has a slope of $+20$ dB/decade

For $G(s) = 3(s + 1)$: ($\omega = 1$ is the break frequency)



Example 3

- Bode plot of

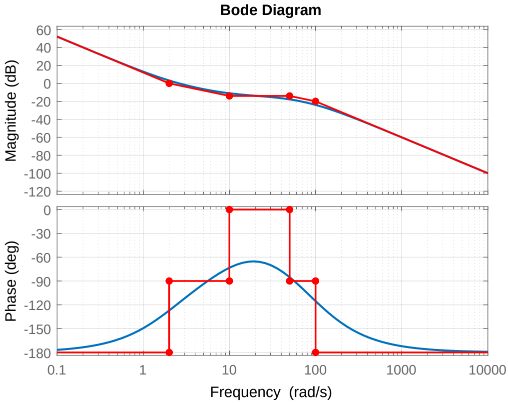
$$G(s) = \frac{4(\frac{1}{2}s+1)(\frac{1}{10}s+1)}{s^2(\frac{1}{50}s+1)(\frac{1}{100}s+1)}$$

Example 3

- Bode plot of

$$G(s) = \frac{4(\frac{1}{2}s+1)(\frac{1}{10}s+1)}{s^2(\frac{1}{50}s+1)(\frac{1}{100}s+1)}$$

- Break frequencies: $\omega = 2$, $\omega = 10$, $\omega = 50$ and $\omega = 100$.



Complex Conjugate Poles or Zeros $(1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2)^{\pm 1}$

- The logarithmic gain and the phase angle are

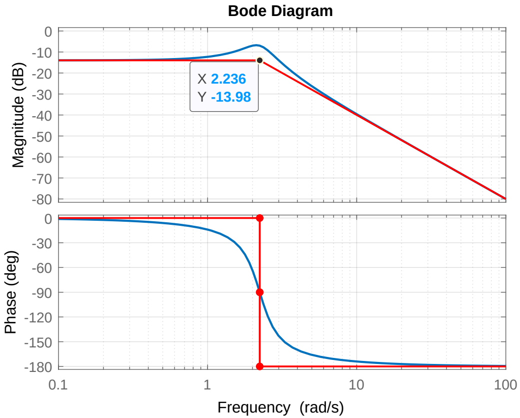
$$|G| = -10\log_{10}\left((1-\frac{\omega^2}{{\omega_n}^2})^2 + 4\zeta^2(\frac{\omega}{\omega_n})^2\right)\ dB$$

$$\phi(\omega) = -\tan^{-1}\frac{2\zeta(\frac{\omega}{\omega_n})}{1-\frac{\omega^2}{{\omega_n}^2}}$$

- **Break Frequency** or **Corner Frequency**: $\omega = \omega_n$
- For $\omega \ll \omega_n$: $|G| = 0$, $\angle G = 0$
- For $\omega \gg \omega_n$: $|G| = -40\log\frac{\omega}{\omega_n}$, $\angle G = -180°$
- The asymptotic curve has a slope of -40 dB/decade
- Phase at break frequency is $-90°$.
- The magnitude asymptotes meet at the 0 dB

Complex Conjugate Poles or Zeros $(1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2)^{\pm 1}$

Bode plot for $G = \frac{1}{s^2+s+5} \rightarrow \omega = \sqrt{5} = 2.23$ is the break frequency and $\zeta = 0.22$

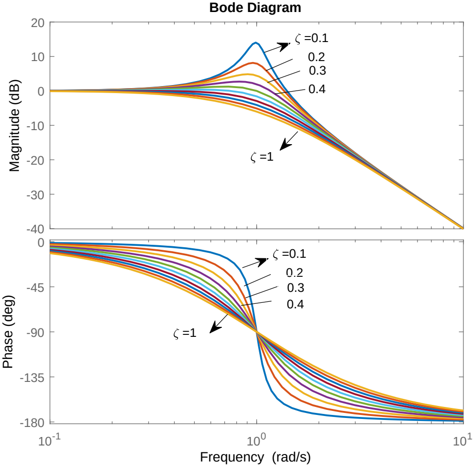


- For $\zeta < 0.7$, there is difference between the actual magnitude curve and the asymptotic approximation, which is a function of the damping ratio.

Resonant Frequency

Resonant frequency: $\omega_r = \omega_n\sqrt{1-2\zeta^2} \quad \zeta < 0.7$

Maximum value of the magnitude $|G(j\omega)|$: $M_{p\omega} = |G(j\omega_r)| = (2\zeta\sqrt{1-\zeta^2})^{-1}$



Example 4

- Bode plot of

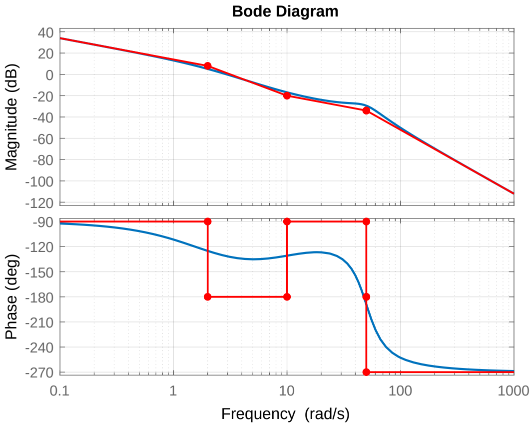
$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)(1 + j0.6(\omega/50) + (j\omega/50)^2)}$$

Example 4

- Bode plot of

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)(1 + j0.6(\omega/50) + (j\omega/50)^2)}$$

- Break frequencies: $\omega = 2$, $\omega = 10$ and $\omega = 50$.



Minimum Phase Functions

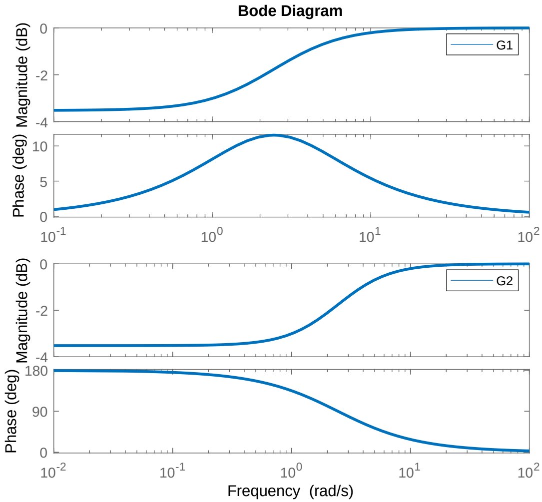
- A transfer function is called a minimum phase transfer function if all its zeros lie in the left-hand s-plane.
- It is called a non-minimum phase transfer function if it has zeros in the right-hand s-plane.
- Two systems $G_1(s) = \frac{s+z}{s+p}$ and $G_2(s) = \frac{s-z}{s+p}$:

  - have the same amplitude characteristics

  - the phase characteristics are different
- Meaning of the term **minimum phase**:

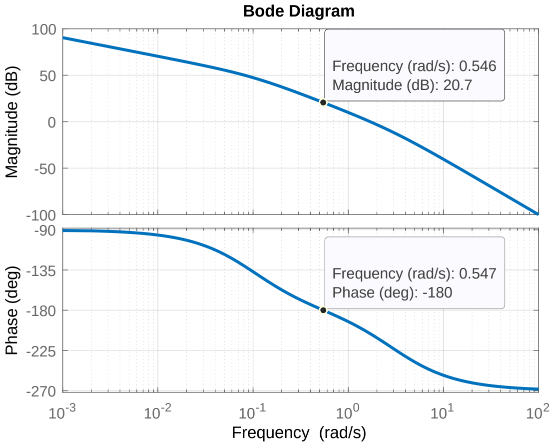
  The range of phase shift of a minimum phase transfer function is the least possible to a given amplitude curve. (*The overall average slope of the lines with the minimum phase system is less than the non-minimum phase system*.)

Minimum Phase Functions



Stability Analysis using Bode Plot

- The closed loops system is stable, if the magnitude at the frequency with phase of -180° satisfy $20\log|GH| < 0$.



Comments on sketching Bode diagrams

- A alternative technique would be to ignore the asymptotes, and just calculate and plot the true gain and phase over a grid of frequencies. This is **not recommended** for a number of reasons:

  1) a lot more points are required to get the same accuracy (particularly when the diagram is to be used for control system analysis and design, as only a small region is required accurately in this case);

  2) the structure of the diagram is then lost. A control engineer will often prefer a good sketch, with the asymptotes shown, to an accurate computer generated diagram – since this gives a better idea of how things can be changed to improve the behavior of the controlled system.

- When a question asks you to “draw” a Bode diagram, it is basically asking you to produce a drawing showing the straight line asymptotes and a rough approximation to the true gain and phase by rounding the corners appropriately.

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