

### Exercise 1

Calculate the Fourier series of the unfiltered output,  $u_{oi}$ , of the buck converter. What are the highest Fourier components in the input and output of the filter. How the variation in the load influences the functioning of the circuit? In the calculation, you may use the function  $h(x)$  (figure 1) which corresponds to the switching with  $D$ , the switch duty ration.  $U_d = 40V$ ,  $U_o = 5V$ ,  $P_o \geq 5W$ ,  $L = 43,75\mu H$ ,  $C = 470\mu F$  and  $f_s = 50kHz$

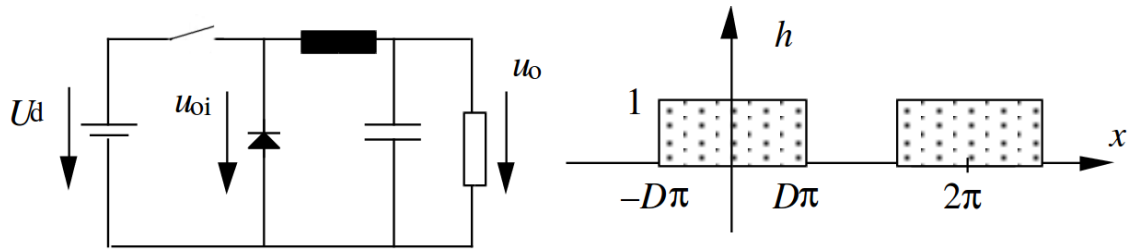


Figure 1: Buck converter and the switching function  $h$ .

### Hints

The switching function  $h$  can be represented with the Fourier series as

$$h(x) = h_0 + \sum_{n=1}^{+\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{1}$$

where the average value is

$$h_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) dx \tag{2}$$

and the coefficients are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x) \cos(nx) dx \tag{3}$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x) \sin(nx) dx \tag{4}$$

The LC-filter (figure 2) transfer function  $G$  is given by

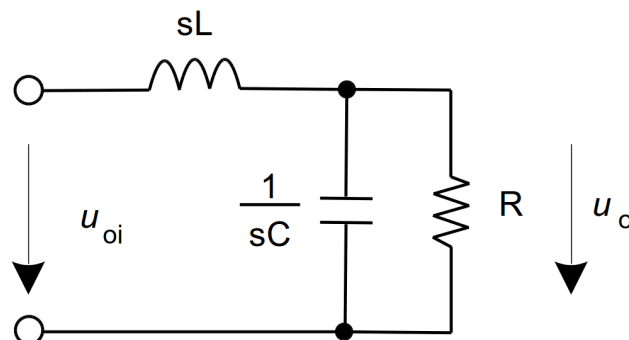


Figure 2: LC-filter.

$$G(s) = \frac{u_o}{u_{oi}} = \frac{\frac{R/sC}{R+1/sC}}{sL + \frac{R/sC}{R+1/sC}} \quad (5)$$

which after few modifications becomes

$$G(s) = \frac{1}{LC \left( s^2 + \frac{1}{RC}s + \frac{1}{LC} \right)} \quad (6)$$

and which can be rewritten as

$$G(s) = \frac{\omega_o^2}{s^2 + 2\xi\omega_o s + \omega_o^2} \quad (7)$$

where the resonance frequency  $\omega_o$  is

$$\omega_o = \frac{1}{\sqrt{LC}} \quad (8)$$

and the damping ratio  $\xi$  (xi) is

$$\xi = \frac{1}{2R} \sqrt{\frac{L}{C}} \quad (9)$$

## Solution

Using the figure 1 we can calculate the Fourier series of  $h$ . Using equation 2, the average value is

$$h_o = \frac{1}{2\pi} \int_{-D\pi}^{D\pi} dx = D \quad (10)$$

Because the function  $h$  is even ( $h(-x) = h(x)$ ), the component  $b_n = 0$  and we just need to calculate the term  $a_n$  using equation 3

$$a_n = \frac{1}{\pi} \int_{-D\pi}^{D\pi} \cos(nx) dx = \frac{2}{n\pi} \sin(nD\pi) \quad (11)$$

and we can write the switching function,

$$h(x) = D + \frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{1}{n} \sin(nD\pi) \cos(nx) \quad (12)$$

we get the voltage  $u_{o1}$  with

$$u_{o1} = h(x)U_d = U_d \left( D + \frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{1}{n} \sin(nD\pi) \cos(nx) \right) \quad (13)$$

The amplitude of the harmonics of  $u_{o1}$  is then given by

$$\hat{u}_{o1} = \frac{2U_d}{n\pi} \sin(nD\pi) \quad (14)$$

with the switching duty ratio being

$$D = \frac{U_o}{U_d} = 0,125 \quad (15)$$

The maximal amplitude of the harmonics of  $u_{o1}$  is when  $n = 1$ . With the numerical values we get

$$\hat{u}_{o1} = \frac{2U_d}{\pi} \sin(D\pi) \approx 9,745V \tag{16}$$

So, the Fourier component at  $f = 50\text{kHz}$  is larger than the voltage DC-component.

Now, lets calculate the effect of the filter on the harmonics. Using the equations 8 and 9 with the numerical values we get the resonance frequency  $\omega_o \approx 6973,68\text{rad.s}^{-1}$  and the damping ratio  $\xi \approx 0,1525/R$ .

From equation 7, we can obtain the poles (denominator is zero) of the transfer function  $G(s)$ . There is no zeros because the numerator is never equal to zero. To obtain the poles we need to solve  $s$  for

$$s^2 + 2\xi\omega_o s + \omega_o^2 = 0 \tag{17}$$

The equation 17 is a second order equation which is solve using

$$s = \frac{-2\xi\omega_o \pm \sqrt{(2\xi\omega_o)^2 - 4\omega_o^2}}{2} = -\xi\omega_o \pm \sqrt{(\xi\omega_o)^2 - \omega_o^2} = \omega_o(-\xi \pm \sqrt{\xi^2 - 1}) \tag{18}$$

If we choose  $R = 1\Omega$ ,  $s \approx -1063,49 \pm 6892,11j$

$R = 100\Omega$ ,  $s \approx -10,63 \pm 68,92j$

$R = 0,1525\Omega$ ,  $s = -6973,68$

$R = 0,152\Omega$ ,  $s = -6996,62 \pm 566,105j$

It can be noticed that the larger the resistor, the more imaginary are the poles. The figure 3 give a visual of the poles in the pole-zero map. To obtain the harmonics after filtering we

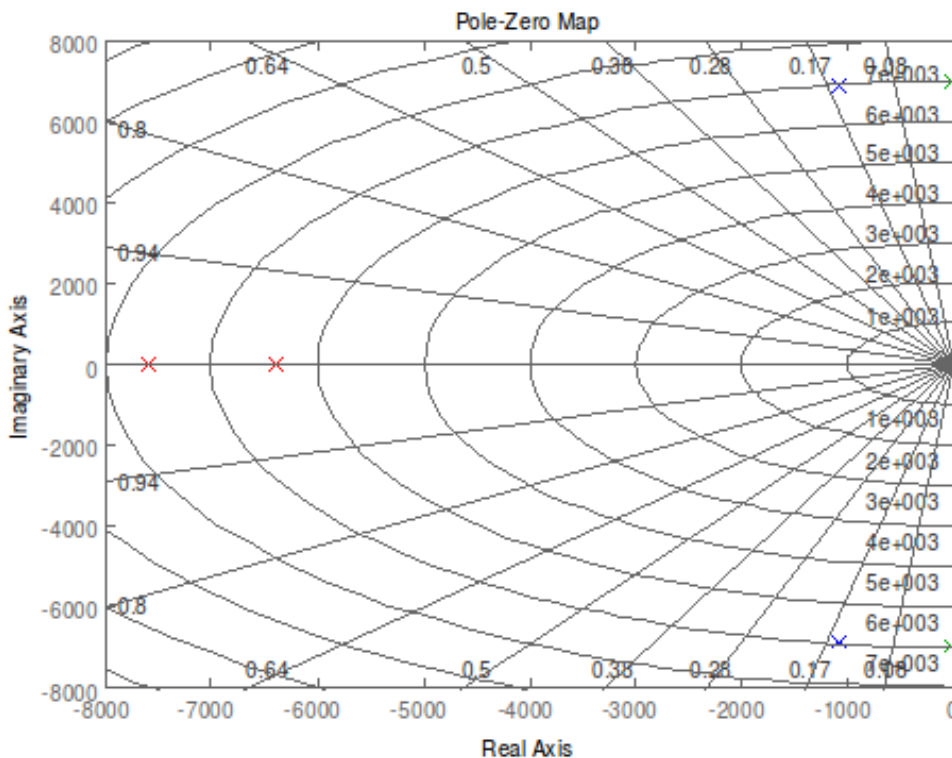


Figure 3: Poles of the LC-filter in the Pole-zero Map.

use equation 7 and replacing  $s$  by  $jn\omega_s$ , we get

$$u_{o,n} = G(s = jn\omega_s)u_{o1,n} = \frac{u_{o1,n}\omega_o^2}{-(n\omega_s)^2 + 2j\xi\omega_on\omega_s + \omega_o^2} \quad (19)$$

the  $n$  in indexes indicates the  $n$ th harmonic. The resonance frequency is usually chosen to be at least one decades smaller than the switching frequency. We can write that  $\omega_o < \omega_s$  and also  $n \geq 1$ ,  $n^2 \geq n$  and  $\xi < 1$  so that the equation 19 can be approximated to

$$u_{o,n} = \frac{u_{o1,n}}{-\left(\frac{n\omega_s}{\omega_o}\right)^2 + 2j\xi\omega_on\frac{\omega_s}{\omega_o} + 1} \approx -\left(\frac{\omega_o}{n\omega_s}\right)^2 u_{oi,n} \quad (20)$$

The minus is due to the  $\pi$  phase shift (see figure 4) that occur when the frequency is higher than the resonance frequency. With the numerical values and the equation 20, we get that

$$\hat{u}_{o1} = 4,8018\text{mV}$$

In the exercise 2 of the exercise session 1, we had that the peak-to-peak value of the output voltage  $\Delta U_{o,max} \approx 10,64\text{mV}$  ( $= 2 \times 5,32\text{mV}$ ). In practice, the output ripple is mostly form by the first harmonic.

In the figure 4 is represented the Bode diagrams for  $R = 1\Omega$ ,  $100\Omega$  and  $0,152\Omega$ . In the Bode

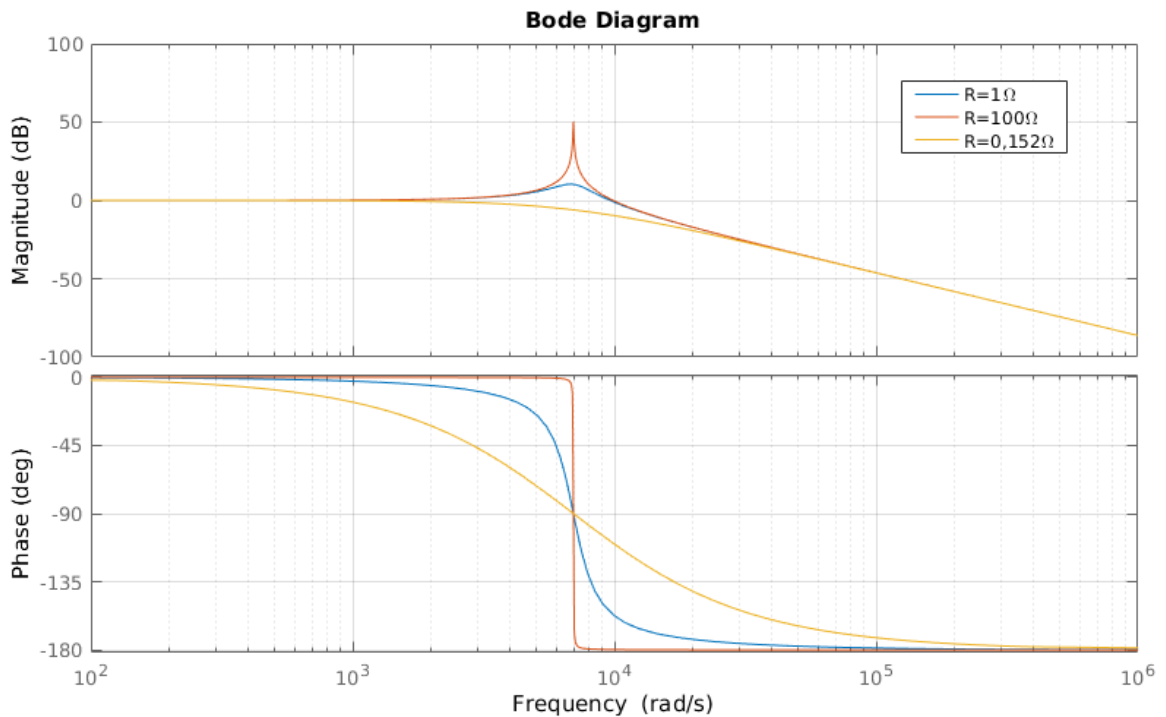


Figure 4: Bode diagrams of the LC-filter.

diagrams the magnitude is defined as

$$|G(j\omega)|_{dB} = 20 \log \left| \frac{U_o(j\omega)}{U_{in}(j\omega)} \right| \quad (21)$$

and the argument is

$$\text{Arg}(G(j\omega)) = \text{Arg} \left( \frac{U_o(j\omega)}{U_{in}(j\omega)} \right) \quad (22)$$

From the Bode-diagrams we see that when the load is small, the amplification becomes larger at the resonance frequency. At the resonance frequency with a resistance of  $100\Omega$  the output voltage can be 200 times larger.

When the load become larger ( $R$  becomes smaller), the amplification becomes smaller. When the resistance become small enough, the output voltage reach zero (short-circuit) The Matlab code to obtain the figures 3 and 4:

```
L = 43.75e-6;
C = 470e-6;
R = 1;
wo = 1/sqrt(L*C);

ksii = 1/(2*R)*sqrt(L/C);
G1 = tf(wo^2,[1 2*ksii*wo wo^2]);
G2 = tf(wo^2,[1 2*ksii*wo/100 wo^2]);
G3 = tf(wo^2,[1 2*ksii*wo/0.152 wo^2]);
figure;
bode(G1, G2, G3);
grid on;
figure;
pzmap(G1, G2, G3);
grid on;
```

## Exercise 2

a) Using figure 5 (first without taking in account  $L_1$  and  $C_1$ ), draw the current  $i_d$ , when  $P_o = 5W$ ,  $U_d = 40V$ ,  $U_o = 5V$ ,  $L = 43,75\mu H$  and  $f_s = 50kHz$ . The output voltage is considered to be constant.

b) Using the complete figure 5, let's consider that the inductance  $L$  is large enough that the current  $i_L$  is constant. Calculate the spectrum of the current  $i_d$ . You can use the same switching function  $h$  (figure 1) of the previous exercise. The peak value of the harmonic current  $\hat{i}_{d1}$  must be limited to  $0,2mA$ . What is the maximum authorized resonance frequency  $\omega_o$ .

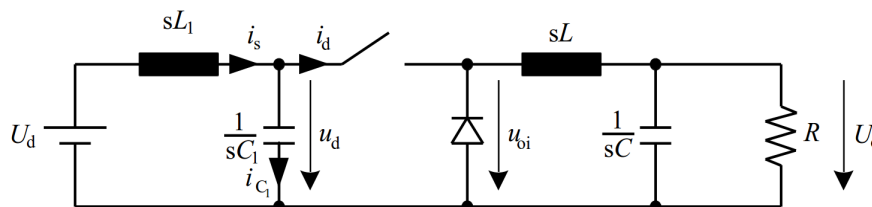


Figure 5: Buck converter.

## Hints

Basically same hints as in the previous exercise.

Solution

Part a

When the switch is conducted, the current in the switch is the same as in the inductance  $L$ . When the switch is open, its current  $i_d = 0A$ . The figure 6 shows the current  $i_d$  waveform. Let's calculate the values of  $i_{Lmin}$  and  $i_{Lpeak}$ . When the switch is conducting, we have

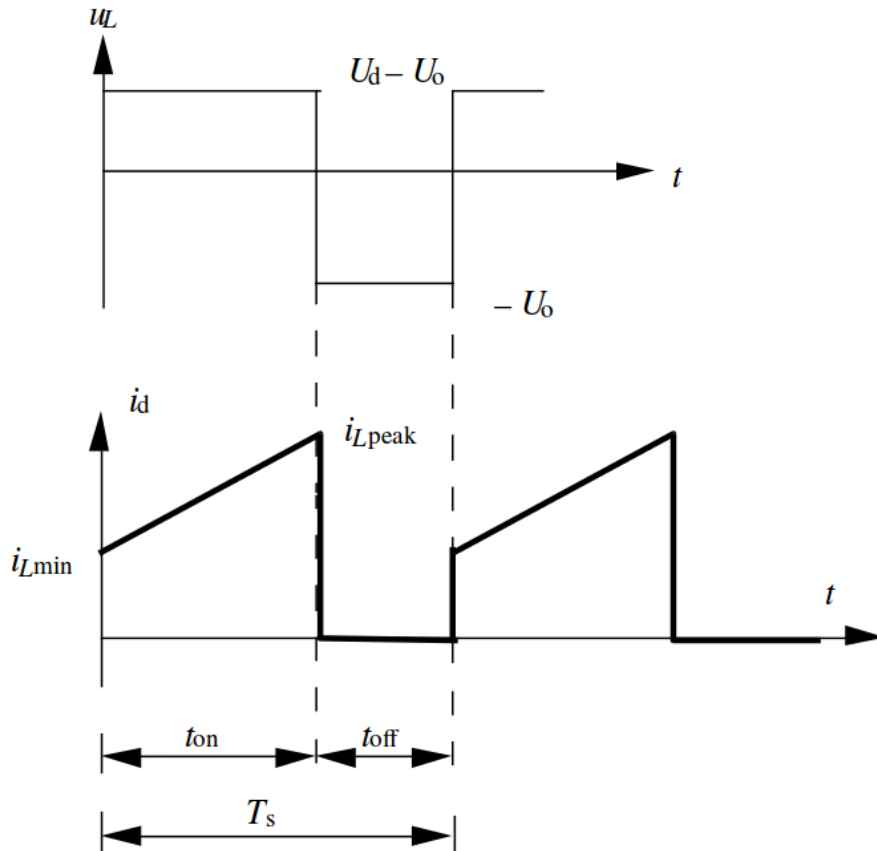


Figure 6: Buck converter waveforms  $u_L$  and  $i_d$ .

$$i_L = i_{Lmin} + \int_0^t u_L dt = i_{Lmin} + \frac{U_d - U_o}{L}t, 0 < t < t_{on} \tag{23}$$

the maximal value is at  $t = t_{on} = DT_s$ .

$$i_{Lpeak} = i_{Lmin} + \frac{U_d - U_o}{L}t, 0 < t < t_{on} \tag{24}$$

When the switch is not conducting, the current in the inductance decreases. When the switch is conducting again the current is  $i_{Lmin}$ . And we get

$$i_{Lmin} = i_{Lpeak} - \frac{U_o}{L}(1 - D)T_s \tag{25}$$

Using the average value  $I_o$  of the current  $i_L$ , we get

$$i_{Lpeak} = i_o + \frac{i_{Lpeak} - i_{Lmin}}{2} = I_o + \frac{U_o}{2L}(1 - D)T_s \tag{26}$$

Since the output power is  $P_o = U_o I_o = 5W$  we get  $I_o = 1A$ . With the equation 26, we get  $i_{Lpeak} = 2A$ . And using equation 25, we get  $i_{Lmin} = 0A$ .

Part b

When the inductance is large enough, the current  $i_d$  remains constant when the switch is conducting. It becomes like in figure 7 Using the Fourier series of the switching function  $h$

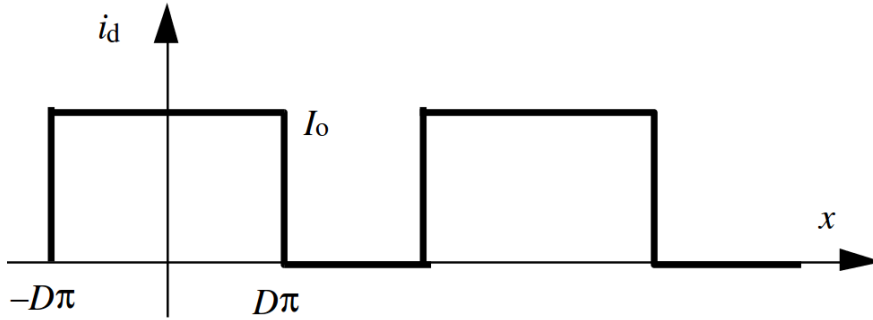


Figure 7: Buck converter waveforms  $i_d$  with large inductance  $L$ .

(equation 12) of the exercise 1, we get

$$i_d(x) = h(x)I_o = I_o \left( D + \frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{1}{n} \sin(nD\pi) \cos(nx) \right) \tag{27}$$

The  $n$ th harmonic component peak value is

$$\hat{i}_{dn} = \frac{2I_o}{n\pi} \sin(nD\pi) \tag{28}$$

where  $D = 5/40$  which gives  $I_d = I_o D = 0,125$ . The largest peak value is obtained when  $n = 1$  and is  $\hat{i}_{d1} \approx 0,243A$ .

The harmonic current at the switching frequency seen from the source has to be limited to 0,2 mA. Therefore, a low-pass filter has to be added on the input side to reduce the amplitude. We can use the schematics shown in figure 8 The voltage of the capacitor  $C_1$  is

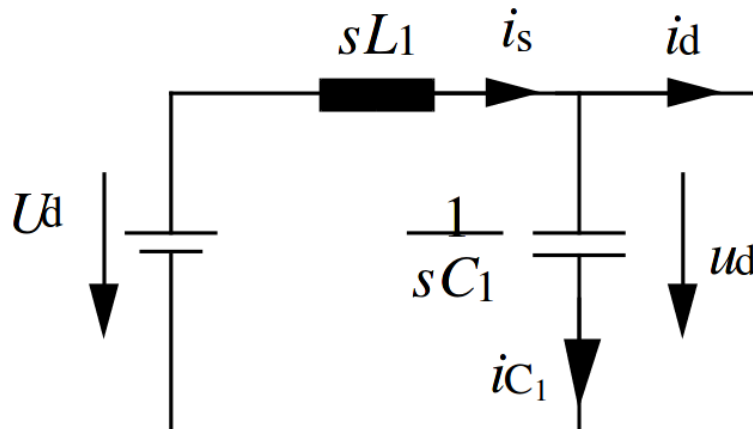


Figure 8: Lowpass filter connected to the input of the buck converter. The current harmonic  $i_{s1}$  must be limited to 0.2 mA.

the input voltage of the buck converter. Its average value is equal to the voltage  $U_d$  because there is no loss taken in account. The average value of the current  $i_s$  is the same as the

average value of  $i_d$ ,  $I_d$ . In AC-analysis  $U_d = 0V$ . The AC-components are marked with a tilde.

$$\tilde{u}_d = \frac{i_{C1}}{sC_1} = -\tilde{i}_s s L_1 \tag{29}$$

and the filter's currents

$$\tilde{i}_s = \tilde{i}_d + i_{C1} = \tilde{i}_d - \tilde{i}_s s^2 L_1 C_1 \tag{30}$$

Which leads to

$$\frac{\tilde{i}_s}{\tilde{i}_d} = \frac{1}{1 + s^2 L_1 C_1} = \frac{1}{1 - \left(\frac{\omega}{\omega_{o1}}\right)^2} \tag{31}$$

where the resonance frequency of the input lowpass filter is

$$\omega_{o1} = \frac{1}{\sqrt{L_1 C_1}} \tag{32}$$

In the equation 32, we can see that  $\omega \gg \omega_{o1}$  is required, because if otherwise the harmonic component of  $i_s$  would be amplified. The minus come from the phase shift at the resonance frequency. On the figure 9, we have the Bode diagram of the equation 32. So negative value means that the current is shifted 180 degree (see Bode diagram). To obtain the magnitude, absolute value is applied.

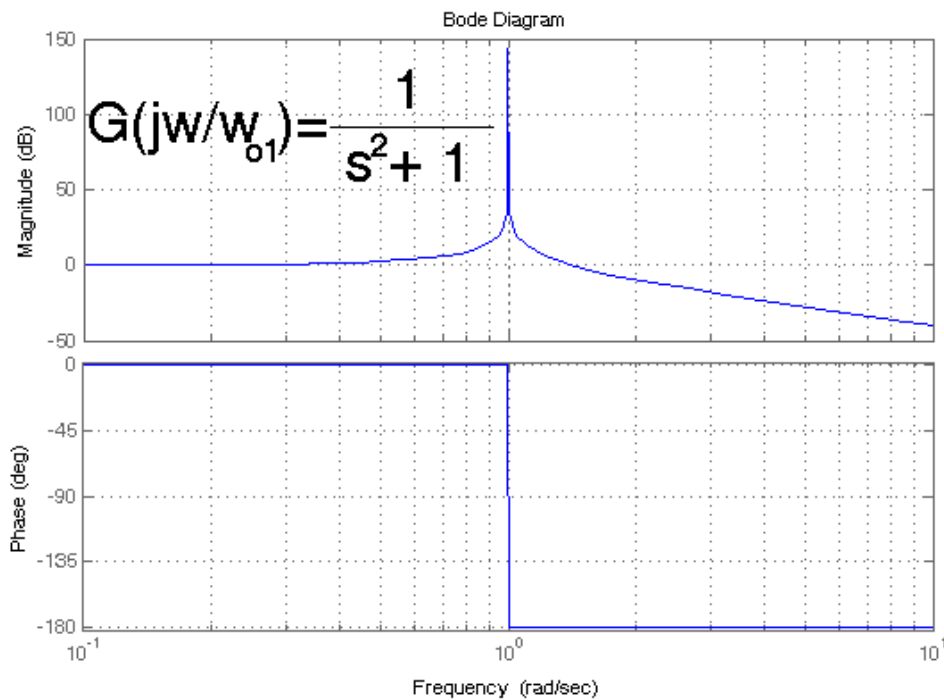


Figure 9: Bode diagram of  $G(s) = \tilde{i}_s/\tilde{i}_d$ .

Using the transfer function 31 for nth harmonic current we get

$$\frac{\hat{i}_{sn}}{\hat{i}_{dn}} = \frac{1}{\left|1 - \left(n\frac{\omega}{\omega_{o1}}\right)^2\right|} = \frac{1}{\left(n\frac{\omega}{\omega_{o1}}\right)^2 - 1}, \quad \omega \gg \omega_{o1}, \quad n = \mathbb{Z}^+ \tag{33}$$



and we obtain the resonance frequency being

$$\omega_{o1} = n\omega_s \sqrt{\frac{1}{1 + \frac{\hat{i}_{dn}}{\hat{i}_{sn}}}} \quad (34)$$

And with the numeric values gives  $\omega_{o1} = 9020$  rad/s which is the maximal resonance frequency allowed to have  $\hat{i}_{s1} \leq 0,2$  mA.

### Exercise 3

A boost converter is represented in the figure 10, plot the current  $i_L, i_k, i_D, i_C$  and  $i_o$  and the voltages  $U_o, U_D$  and  $U_L$ .

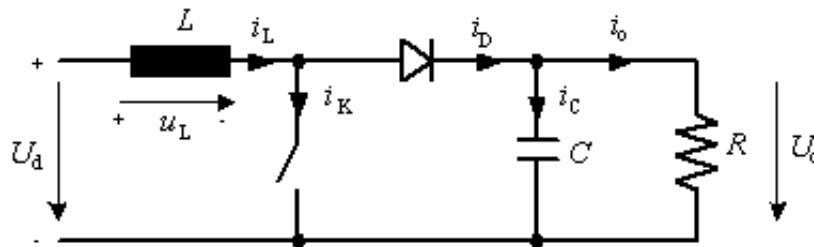


Figure 10: Boost converter.

What is the relation between average output current  $i_{oB}$  and the average input current  $i_{LB}$ ? The output voltage of the boost converter  $U_o = 24V$  is set to be constant with a duty ratio  $D$ . The input voltage is limited with  $8V \leq U_d \leq 16V$ . The output power is  $P_o \geq 5W$ , the switching frequency is  $f_s = 20kHz$ ,  $C = 470\mu F$ . Calculate the value of the choke  $L$  that keeps the circuit in continuous conduction mode (CCM).

### Solution

The waveforms are given in the figure 11 Using the figure

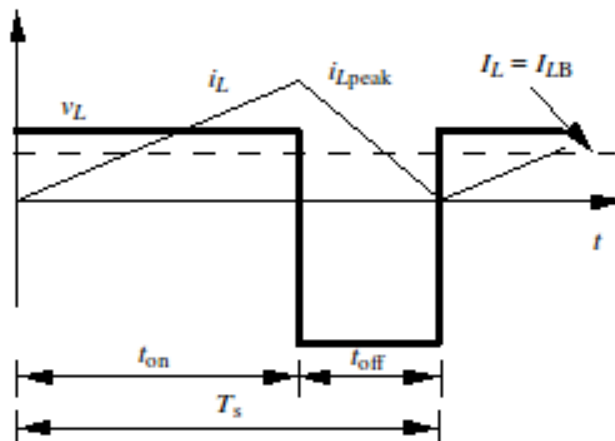


Figure 12: Current waveforms  $i_L$  and  $u_L$  of a boost converter at the limit of CCM.

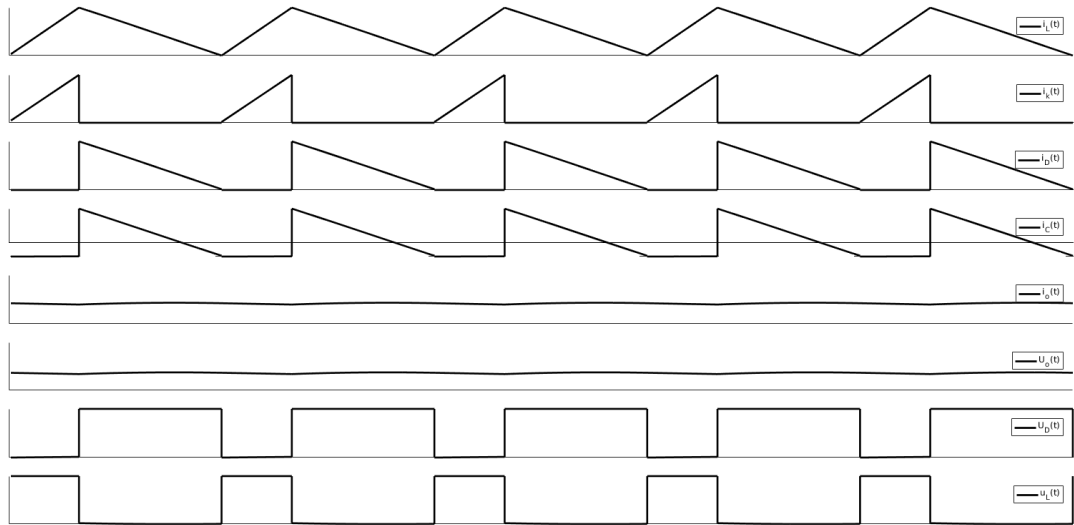


Figure 11: Waveforms of boost converter.

With some geometry we can calculate the average current  $I_{LB}$  and we get

$$I_{LB} = \frac{i_{Lpeak}}{2} = \frac{U_d}{2L} t_{on} = \frac{T_S U_o}{2L} D(1 - D) \quad (35)$$

Because there are no loss in the circuit we can write  $U_d I_L = U_o I_o$ . In a boost converter we have

$$\frac{I_o}{I_L} = \frac{U_d}{U_o} = 1 - D \quad (36)$$

and we get

$$I_{oB} = I_{LB}(1 - D) = \frac{T_S U_o}{2L} D(1 - D)^2 \quad (37)$$

If the average output current is lower than  $I_{oB}$ , the circuit enters in discontinuous conduction mode (DCM). We have to calculate the value of  $D$  which gives the maximal value of  $I_{oB}$ . Using equation 37, we can obtain the maximal value of  $D$ .

$$\frac{dI_{oB}}{dD} = \frac{T_S U_o}{2L} \frac{d}{dD} (D(1 - D)^2) = 0 \quad (38)$$

which leads to

$$3D^2 - 4D + 1 = 0 \quad (39)$$

and the solutions are  $D = 1$  or  $D = 1/3$ . Only the solution  $D = 1/3$  is valid. Using that value is the equation 37, we obtain

$$I_{oB,max} = \frac{2T_S U_o}{27L_{min}} \Rightarrow L_{min} = \frac{2T_S U_o}{27I_{oB,max}} \quad (40)$$

Since  $P_o \geq 5W$  and  $U_o = 24V$ , the average output current is  $I_o = 5/24A$ . Using 40, we finally obtain the minimum value of the choke is  $L_{min} = 0,427mH$ .