Advanced microeconomics 3: game theory
Spring 2023

Problem set 4 (Due 27.2.2023)

1. Two firms decide simultaneously whether or not to invest in a natural monopoly market. If only one firm invests, that firm gets payoff 1. If both firms invest, they both get payoff -1. A firm that does not invest gets 0.

    (a) Formalize and analyze this as a strategic form game.

    (b) Add some incomplete information to the investment payoffs of the firms and analyze the resulting game of incomplete information. Show that you can purify the mixed strategy equilibrium in (a).

2. Consider private provision of public goods with incomplete information. The two firms choose simultaneously whether or not to contribute to the public good. The payoffs are:

	Contribute	Do not contribute
Contribute	$1-\theta_1, 1-\theta_2$	$1-\theta_1, 1$
Do not contribute	$1, 1-\theta_2$	$0, 0$

   where θ_i, $i = 1, 2$, are private cost parameters that are independently distributed according to some atomless distribution $F(\cdot)$ with support $[\underline{\theta}, \overline{\theta}]$. Assume that $\underline{\theta} < 1$.

    (a) Show that best-responses always take the cut-off form.

    (b) Derive a condition for the cut-off that characterizes a symmetric Bayesian equilibrium of the game and solve it explicitly when θ_i are uniformly distributed over $[0, 2]$.

    (c) Suppose that $\underline{\theta} > 1 - F(1)$. Can you now find asymmetric equilibria in addition to the symmetric one?

3. Consider the following common values auction. There are two bidders whose types θ_i are independently drawn from a uniform distribution

on $[0, 100]$. The value of the object to both bidders is the sum of the types, i.e. $\theta_i + \theta_j$. The object is offered for sale in a first price auction. Hence the payoffs depend on the bids b_i and types as follows (we ignore ties for convenience):

$$u_i(b_i, b_j, \theta_i, \theta_j) = \begin{cases} \theta_i + \theta_j - b_i \text{ if } b_i > b_j, \\ 0 \text{ otherwise.} \end{cases}$$

(a) Show by a direct computation that the linear strategies where $b_i = \theta_i$ for $i = 1, 2$ form a Bayesian equilibrium in this game.

(b) If $\theta_i = 1$, the equilibrium bid is 1, but it might seem that the expected value of the object is 1+50=51. Why doesn't the bidder behave more aggressively?

(c) Analyze the game above as a second price auction. Does the game have a dominant strategy equilibrium? Find a Bayesian Nash equilibrium of the game. (Hint: Think carefully about the event where changing one's own bid changes one's payoff. What does this imply about the bid of the other player? In symmetric equilibrium, what does this imply about the type of the other player? Alternatively, you may use the guess and verify method of the previous question and verify that a linear symmetric equilibrium exists.).

4. Let us continue with the game of private provision of public goods of Problem 2. The stage game is otherwise as in Problem 2, but we specify here that the private costs θ_i are uniformly distributed over $[0, 2]$. The aim of this excercise is to find a symmetric perfect Bayesian equilibrium of a twice repeated version of the game. The players first choose simultaneously whether or not to contribute in the first period. Then, after observing each others' actions, they choose simultaneously whether or not to contribute in the second period. Both players maximize the sum of payoffs over the two periods.

(a) Argue that if there is a symmetric equilibrium strategy profile, then there must be some cutoff type $\widehat{\theta} \in (0,1)$ such that i contributes in the first period if and only if $\theta_i \leq \widehat{\theta}$.

(b) Suppose that i contributes in the first period if and only if $\theta_i \leq \widehat{\theta}$, where $\widehat{\theta} \in (0,1)$, $i = 1,2$. Derive the posterior beliefs of the players in all information sets of the second period.

(c) Solve the second-period equilibrium if neither player contributed in the first period.

(d) Solve the second-period equilibrium if both players contributed in the first period.

(e) Solve the second-period equilibrium if one player contributed and the other did not contribute in the first period.

(f) Using the continuation payoffs for the second period derived above, solve for the cutoff $\widehat{\theta}$ such that a player with $\theta_i = \widehat{\theta}$ is indifferent between contributing and not contributing in the first period. Argue that you have derived a symmetric perfect Bayesian equilibrium of the game.

(g) Is $\widehat{\theta}$ lower or higher than the corresponding equilibrium cutoff of the one-period version of the game? Discuss the intuition for this result.

5. The standard "stag-hunt" game is defined as follows (https://en.wikipedia.org/wiki/Stag_hunt):

	Hunt stag	Hunt rabbit
Hunt stag	$2,2$	$0,1$
Hunt rabbit	$1,0$	$1,1$

We add here some incomplete information to the game. Assume that with probablity p, a player is a "stag"-type and wants to hunt the stag no matter what the other player does (she does not like rabbit, for example). Similarly with probability q, a player is a "rabbit"-type and has a dominant strategy to hunt the rabbit. With probability $1-p-q$, a player has the preferences described in the table above (call this the "normal"-type). Assume that $2p > 1-q$ and $2q > 1-p$.

(a) Show that if $\max(p, q) > 1/2$, then there is a unique Bayesian equilibrium, otherwise there are multiple Bayesian equilibria.

(b) Consider a two-period game, where the above stag-hunt game with incomplete information is repeated twice. Show that in any perfect Bayesian equilibrium of such a two-period game, the second period behavior of the normal-type is uniquely pinned down by the behavior of the other player in the first period.

(c) Assume that $\max(p, q) < 1/2$ and $p < \alpha < q$, where $\alpha := \frac{1+2p}{4}$. Assume also that the normal-type players maximize the sum of payoffs across the periods (the stag-type hunts stag in both periods, and a rabbit-type hunts rabbit in both periods in any case). Show that there is a symmetric perfect Bayesian equilibrium, where each player hunts the stag with total probability α in the first period. Note that in this equilibrium the normal-type is willing to sacrifice some short-run utility to build a "reputation". Are there other perfect Bayesian equilibria?