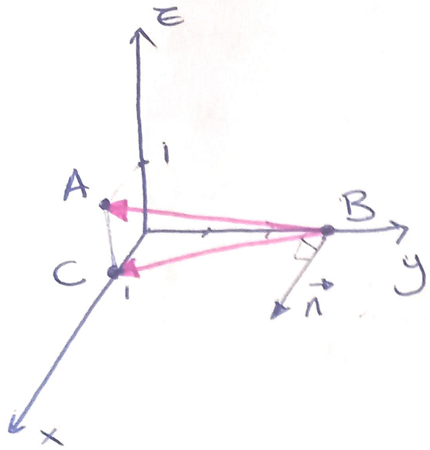
(1)



$A(1,0,1)$ $B(0,3,0)$ $C(1,0,0)$

Consider the vectors

$\vec{BA} = \langle 1,-3,1 \rangle$ and $\vec{BC} = \langle 1,-3,0 \rangle$

A normal vector to the plane is

$$\vec{n} = \vec{BA} \times \vec{BC} = \begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 1 & -3 & 0 \end{vmatrix} = \langle 3,1,0 \rangle$$

Using B as a point on the plane, we get the equation:

$3(x-0)+1(y-3)=0$; $\boxed{3x+y-3=0}$

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(2) $r'(1)$ and $s'(1)$ are tangent vectors when $t=1$:

$$v_1 = r'(1) = (2t, 3t^2, 6t^2)\Big|_{t=1} = (2,3,6)$$

$$v_2 = s'(1) = (2t, 1, 2)\Big|_{t=1} = (2,1,2)$$

Now, we use the formula $v_1 \cdot v_2 = \|v_1\| \cdot \|v_2\| \cdot \cos\alpha$

~~where~~ where α is the angle between the two tangent vectors v_1, v_2.

Then,
$$\cos\alpha = \frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|} = \frac{4+3+12}{\sqrt{4+9+36} \cdot \sqrt{4+1+4}} = \frac{19}{21}$$

$$\Rightarrow \boxed{\alpha = \arccos\left(\frac{19}{21}\right)}$$

3

$r(t) = (a\cos^3(t), a\sin^3(t))$ $\quad a \in \mathbb{R}^+ \quad 0 \leq t < 2\pi$

Using the symmetries of the curve, we can compute the length ℓ as 4 times the length of the curve from $t=0$ to $t=\frac{\pi}{2}$.

That is,

$$\ell = 4\int_0^{\pi/2} |r'(t)|dt$$

We have that:

$$r'(t) = (-3a\cos^2(t)\sin(t), 3a\sin^2(t)\cos(t))$$

and

$$|r'(t)| = \sqrt{9a^2\cos^4(t)\sin^2(t) + 9a^2\sin^4(t)\cos^2(t)} =$$

$$= 3a\sqrt{\cos^2(t)\sin^2(t)(\underbrace{\cos^2(t)+\sin^2(t)}_{1})} = 3a|\cos(t)\sin(t)|$$

If $t \in [0, \frac{\pi}{2}]$ then $|\cos(t)\sin(t)| = \cos(t)\sin(t) = \frac{1}{2}\sin(2t)$

Therefore, $|r'(t)| = \frac{3a}{2}\sin(2t)$

Finally,

$$\ell = 4\int_0^{\pi/2} \frac{3a}{2}\sin(2t)dt = 6a\int_0^{\pi/2}\sin(2t)dt =$$

$$= -3a\cos(2t)\Big]_0^{\pi/2} = 6a$$

We want $\ell = 30$. Thus, $\boxed{a=5}$.

(4)

(b) Parametric equation: $r(t) = (x(t), y(t), z(t))$

The projection of the curve onto the xy-plane is a circle:

$$x(t) = \cos(t) \qquad 0 \leq t < 2\pi$$
$$y(t) = \sin(t)$$

Using the equation of the plane $x + 2y + z = 0$ we obtain:

$$x(t) + 2y(t) + z(t) = 0 \Rightarrow z(t) = -\cos(t) - 2\sin(t)$$

$$\Rightarrow \boxed{r(t) = (\cos(t), \sin(t), -\cos(t) - 2\sin(t))}$$

(c) Tangent line to the curve at $(0, 1, -2)$

First, we find t such that $r(t) = (0, 1, -2)$

Note that $\sin(t) = 1 \Leftrightarrow t = \frac{\pi}{2}$

In addition, $r(\pi/2) = (0, 1, -2)$

Tangent vector:

$$r'(t) = (-\sin(t), \cos(t), \sin(t) - 2\cos(t))$$

At $(0, 1, -2)$:

$$r'(\pi/2) = (-1, 0, 1)$$

Equation of the tangent line:

$$\ell(t) = (0, 1, -2) + t(-1, 0, 1) = (-t, 1, t-2) \quad t \in \mathbb{R}.$$

$with(plots):$

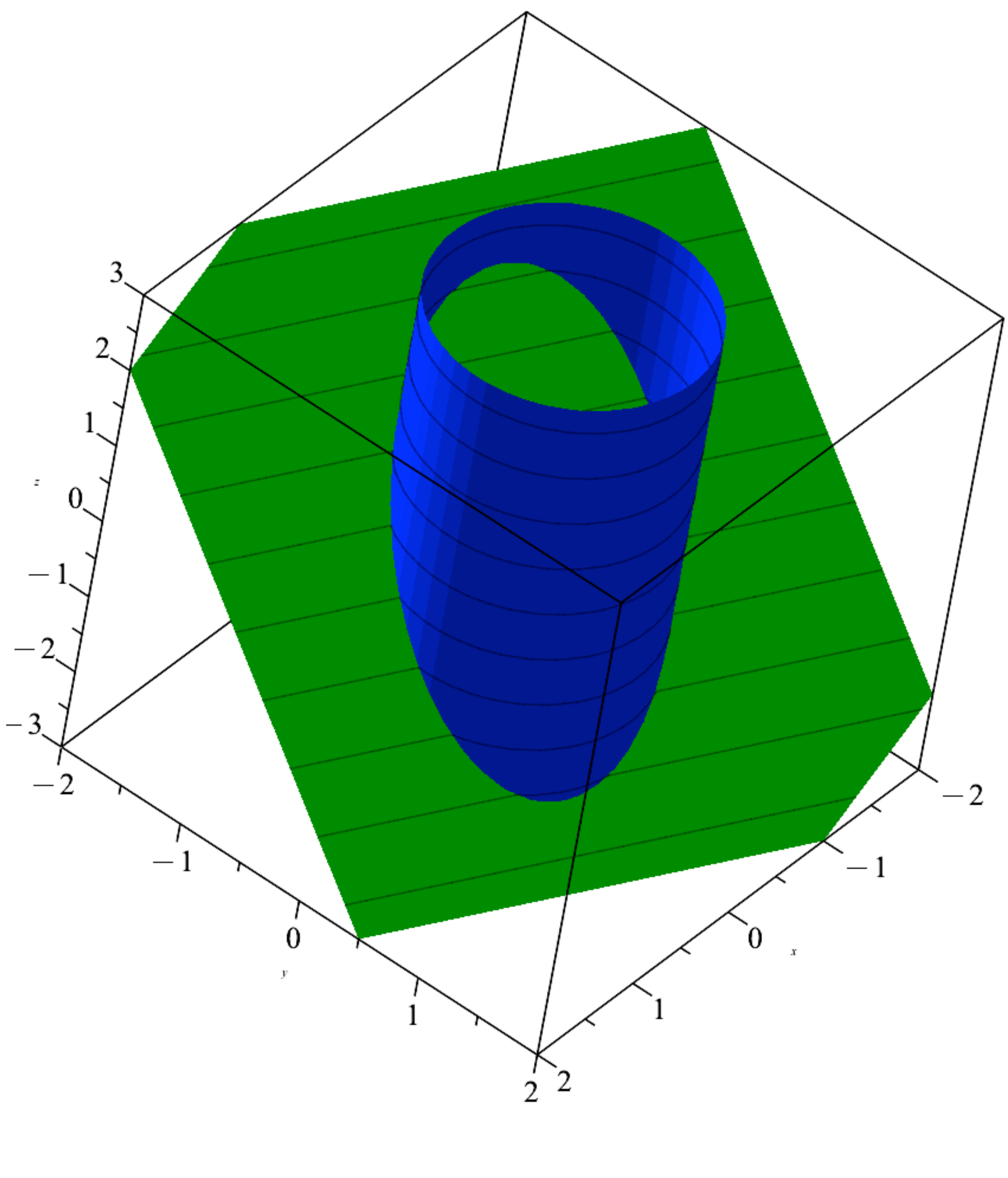
$f := x^2 + y^2 - 1$

$$f := x^2 + y^2 - 1 \quad (1)$$

$g := x + 2\,y + z$

$$g := x + 2\,y + z \quad (2)$$

$implicitplot3d([f, g], x = -2..2, y = -2..2, z = -3..3, color = [blue, green])$



5

No, it is not possible for two level curves to intersect. By contradiction, if $f(x,y)=a$ and $f(x,y)=b$ intersect then there is a point (x_0, y_0) such that $f(x_0,y_0)=a$ and $f(x_0,y_0)=b$. But this is impossible since f is a function and, therefore, for a given input (x_0,y_0) there can only be one output. That is, we must have $a=b$.