1

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+2y}{x^2+y^2}$

If $x=0$ then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+2y}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{2}{y} = \infty$

If $y=0$ then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+2y}{x^2+y^2} = \lim\limits_{x\to 0} x = 0$

We got different limits along different paths

$\Rightarrow$ the limit does not exist.

(Other path:

$y=x^2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+2y}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^3+2x^2}{x^2+\cancel{x^4}} = \lim\limits_{x\to 0} \dfrac{x+2}{1+x^2} = 2$)

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y\sqrt{x}}{\sqrt{x^2+y^2}}$

$0 \le \left| \dfrac{y\sqrt{x}}{\sqrt{x^2+y^2}} \right| = \left| \sqrt{\dfrac{y^2 x}{x^2+y^2}} \right| \le |\sqrt{x}|$ since $\dfrac{y^2}{x^2+y^2} \le 1$

$\lim\limits_{(x,y)\to(0,0)} |\sqrt{x}| = 0 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{y\sqrt{x}}{\sqrt{x^2+y^2}} = 0$

By squeeee th.

② Continuity

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

If $(a,b) \neq (0,0)$ the function is continuous.

Let us study the continuity at $(0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0\ ?$$

- $x=0 \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = 0$
- $y=x \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4+0} = 1$
- $y=2x \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{4x^4}{4x^4+x^2} = \lim_{x\to 0} \frac{4x^2}{4x^2+1} = 0$

We obtain different limits along different paths

$\Rightarrow$ the limit does not exist.

$\Rightarrow f$ is not continous at $(0,0)$.

Therefore, the function is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$.

③ $f(x,y) = \ln(x^2+y^2)$ at $(1,-2)$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \quad ; \quad \frac{\partial f}{\partial x}(1,-2) = \frac{2}{5}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} \quad ; \quad \frac{\partial f}{\partial y}(1,-2) = -\frac{4}{5}$$

We consider the vectors:

$u = \langle 1, 0, \frac{2}{5} \rangle$ and $v = \langle 0, 1, -\frac{4}{5} \rangle$

Then, a normal vector to the surface is:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2/5 \\ 0 & 1 & -4/5 \end{vmatrix} = \langle -\frac{2}{5}, \frac{4}{5}, 1 \rangle$$

Or directly with the formula $(\vec{n} = \langle -\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1 \rangle)$.

The tangent plane equation is:

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Thus, the eq. of the tangent plane is:

$$\boxed{z = \ln(5) + \tfrac{2}{5}(x-1) - \tfrac{4}{5}(y+2)}$$

④ $z = \sqrt{x^2+y^2}$

By Chain rule:

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} \\ \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}} \end{cases}$$

Then,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x \cdot \frac{x}{\sqrt{x^2+y^2}} + y \cdot \frac{y}{\sqrt{x^2+y^2}} = \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{\frac{(x^2+y^2)^2}{x^2+y^2}} = \sqrt{x^2+y^2} = z$$