1

$$f(x,y) = e^{x^2+y^2}$$

(a) f has a minimum at $(0,0)$

First, we compute all the needed derivatives:

$$\frac{\partial f}{\partial x} = e^{x^2+y^2} \cdot 2x$$

$$\frac{\partial f}{\partial y} = e^{x^2+y^2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x^2+y^2} \cdot 4x^2 + e^{x^2+y^2} \cdot 2$$

$$\frac{\partial^2 f}{\partial y^2} = e^{x^2+y^2} \cdot 4y^2 + e^{x^2+y^2} \cdot 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{x^2+y^2} \cdot 4xy = \frac{\partial^2 f}{\partial x \partial y}$$

$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0 \Rightarrow (0,0)$ is a critical point (and it is the only one)

Consider the Hessian matrix of f at $(0,0)$:

$$H(0,0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial y \partial x}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The determinant of $H(0,0)$ is $4 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0) > 0$

$\Rightarrow (0,0)$ is a local minimum.

(b) Linear approximation at (0,0)

$$L(x,y) = f(0,0) + \underbrace{\frac{\partial f}{\partial x}(0,0)}_{0}(x-0) + \underbrace{\frac{\partial f}{\partial y}(0,0)}_{0}(y-0)$$

$$\Rightarrow \boxed{L(x,y) = 1}$$

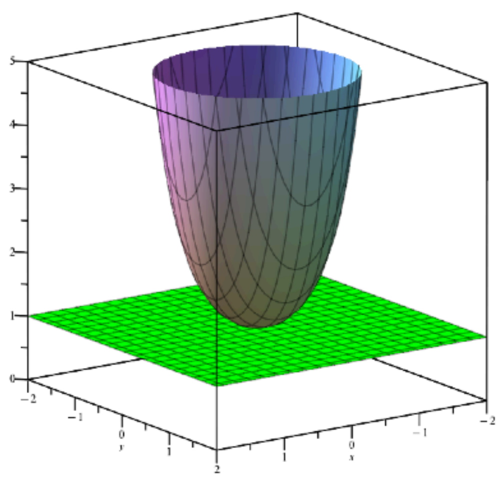
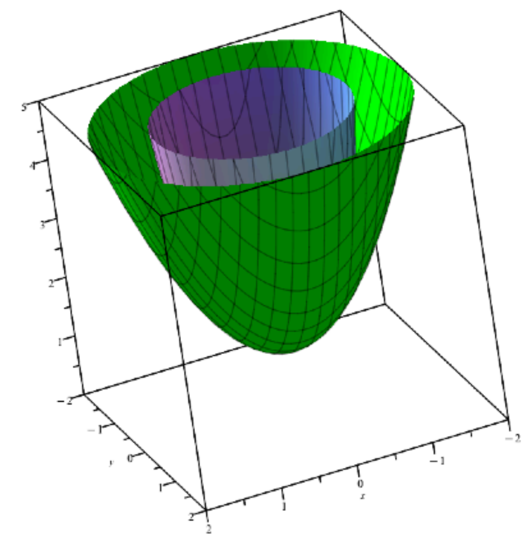
(c) 2-nd degree Taylor polynomial

$$\boxed{Q(x,y)} = L(x,y) + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} =$$

$$= 1 + \frac{1}{2}(2x^2 + 2y^2) = \boxed{1 + x^2 + y^2}$$

$f := e^{x^2+y^2}$

$$f := e^{x^2+y^2} \qquad (1)$$

Linear approximation	**2nd-degree Taylor polynomial**
$s := plot3d(f, x=-2..2, y=-2..2, grid=[40,40], view=0..5):$ $l := plot3d(1, x=-2..2, y=-2..2, grid=[40,40], view=0..5, color=green):$ $display(s, l)$ 	$s := plot3d(f, x=-2..2, y=-2..2, grid=[40,40], view=0..5):$ $q := plot3d(1+x^2+y^2, x=-2..2, y=-2..2, grid=[40,40], view=0..5, color=green):$ $display(s, q)$ 

(2) Critical points of

$$f(x,y) = xye^{-x^2-y^4}$$

First, we compute all the needed derivatives:

$$\frac{\partial f}{\partial x} = ye^{-x^2-y^4} - 2x^2ye^{-x^2-y^4}$$

$$\frac{\partial f}{\partial y} = xe^{-x^2-y^4} - 4xy^4e^{-x^2-y^4}$$

$$\frac{\partial^2 f}{\partial x^2} = -2xye^{-x^2-y^4} - 4xye^{-x^2-y^4} + 4x^3ye^{-x^2-y^4} =$$

$$= -6xye^{-x^2-y^4} + 4x^3ye^{-x^2-y^4}$$

$$\frac{\partial^2 f}{\partial y^2} = -4xy^3e^{-x^2-y^4} - 16xy^3e^{-x^2-y^4} + 16xy^7e^{-x^2-y^4} =$$

$$= -20xy^3e^{-x^2-y^4} + 16xy^7e^{-x^2-y^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{-x^2-y^4} - 4y^4e^{-x^2-y^4} - 2x^2e^{-x^2-y^4} +$$

$$+ 8x^2y^4e^{-x^2-y^4} = \frac{\partial^2 f}{\partial x \partial y}$$

Critical points satisfy the system of equations:

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{array}\right\} \left.\begin{array}{l} ye^{-x^2-y^4} - 2x^2ye^{-x^2-y^4} = 0 \\ xe^{-x^2-y^4} - 4xy^4e^{-x^2-y^4} = 0 \end{array}\right\}$$

Since $e^{-x^2-y^4} \neq 0$ for all (x,y), the system of equations is equivalent to:

$$\left.\begin{array}{l} y - 2x^2y = 0 \\ x - 4xy^4 = 0 \end{array}\right\} \begin{array}{l} \rightarrow y(1-2x^2) = 0 \quad (I) \\ \rightarrow x(1-4y^4) = 0 \quad (II) \end{array}$$

- If $y=0$ then eq. (I) holds and, by (II), $x=0 \rightarrow \boxed{(0,0)}$
- If $1-2x^2=0$, $x = \pm\frac{1}{\sqrt{2}}$ then eq (I) holds and, by (II),

$1-4y^4=0$; $y = \pm\frac{1}{\sqrt{2}} \rightarrow \boxed{\begin{array}{l} (1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2}) \\ (-1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2}) \end{array}}$

2-nd derivative test:

Consider $H(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{bmatrix}$

- $H(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\det(H(0,0)) < 0 \Rightarrow (0,0)$ is a saddle point.

- $H\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = H\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \begin{bmatrix} -2e^{-3/4} & 0 \\ 0 & -4e^{-3/4} \end{bmatrix}$

$\det H > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are local maximum.

- $H\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = H\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \begin{bmatrix} 2e^{-3/4} & 0 \\ 0 & 4e^{-3/4} \end{bmatrix}$

$\det H > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0 \Rightarrow (1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$ are local minimum

$f := x \cdot y \cdot e^{-x^2 - y^4}$

$$f := x\, y\, e^{-y^4 - x^2} \tag{1}$$

$d := plot3d(f, x = -3\,..3, y = -3\,..3, grid = [40, 40], view = -0.4\,..0.4)$