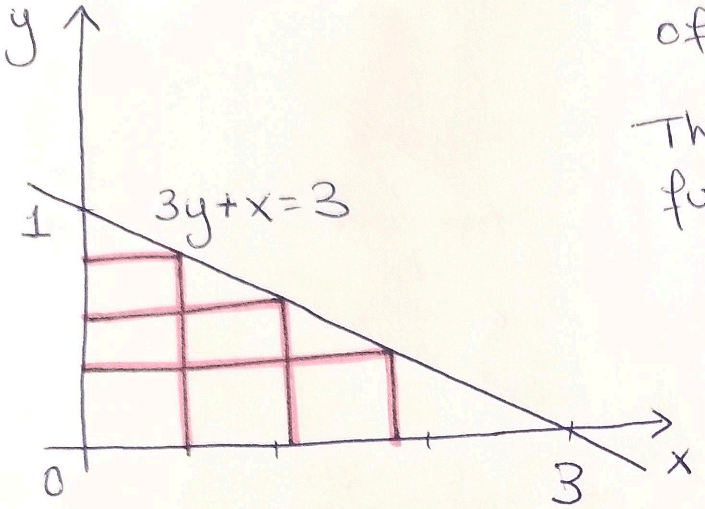
(1)



What is the rectangle of maximum area?

The area is given by the function

$$A(x,y) = xy$$

Therefore, we have to find the maximum of $A(x,y)$ subject to the constraint $3y+x=3$.

Let us apply the method of <u>Lagrange multipliers</u>:

Consider $g(x,y) = 3y+x$. Then, we have to solve the system of equations:

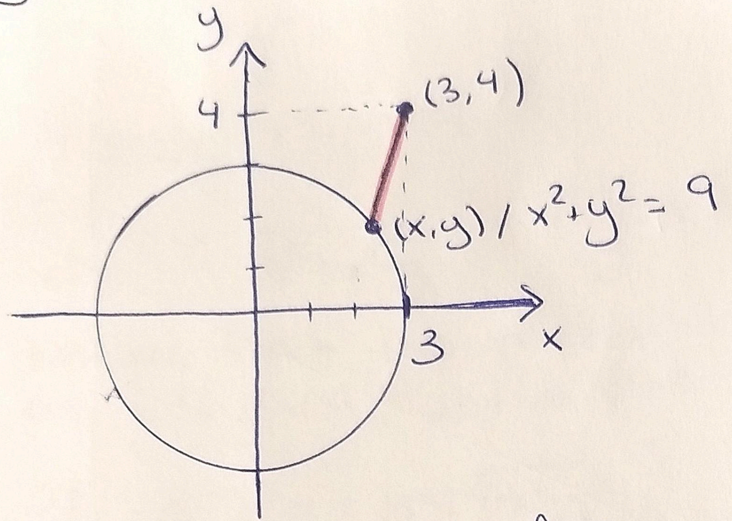
$$\left.\begin{array}{l} \vec{\nabla} A(x,y) = \lambda \vec{\nabla} g(x,y) \\ g(x,y) = 3 \end{array}\right\} \left\{\begin{array}{l} \langle y, x \rangle = \lambda \langle 1, 3 \rangle \\ 3y + x = 3 \end{array}\right\}$$

$$\left.\begin{array}{l} y = \lambda \\ x = \lambda 3 \\ 3y+x=3 \end{array}\right\} \begin{array}{l} \rightarrow x = 3y \\ \rightarrow 3y+x=3 \end{array} \Big\} \; 6y=3 \; ; \boxed{y = \frac{1}{2}} \Rightarrow \boxed{x = \frac{3}{2}}$$

The dimensions for the rectangle of max. area:

- $\boxed{\frac{3}{2} \times \frac{1}{2}}$

②



we have to find the maximum and minimum values of the function

$f(x,y) = (x-3)^2 + (y-4)^2$

subject to

$g(x,y) = 9$ where

$g(x,y) = x^2 + y^2$

By applying the method of Lagrange multipliers.

Remark: the distance from the point (3,4) is $\sqrt{(x-3)^2+(y-4)^2}$ but it is sufficient to consider the square of the distance

We have to solve the system of equations:

$$\left.\begin{array}{l}\nabla f(x,y) = \lambda \nabla g(x,y)\\ g(x,y) = 9\end{array}\right\} \left.\begin{array}{l}\langle 2x-6, 2y-8\rangle = \lambda\langle 2x, 2y\rangle\\ x^2+y^2=9\end{array}\right\}$$

$$\left.\begin{array}{l}2x-6 = 2\lambda x\\ 2y-8 = 2\lambda y\\ x^2+y^2 = 9\end{array}\right\} \xrightarrow[\text{solutions}]{} \left(-\frac{9}{5}, -\frac{12}{5}\right) \text{ and } \left(\frac{9}{5}, \frac{12}{5}\right)$$

Maximum value: $f\left(-\frac{9}{5}, -\frac{12}{5}\right) = 64 \Rightarrow \left(-\frac{9}{5}, -\frac{12}{5}\right)$ farthest point.

Minimum value: $f\left(\frac{9}{5}, \frac{12}{5}\right) = 4 \Rightarrow \left(\frac{9}{5}, \frac{12}{5}\right)$ closest point.

③ Absolute extrema of $f(x,y) = xy + \sqrt{9-x^2-y^2}$ on $x^2+y^2 \leq 9$

1. <u>Interior</u>: $x^2+y^2<9$

$$\frac{\partial f}{\partial x} = y - \frac{x}{\sqrt{9-x^2-y^2}} \quad ; \quad \frac{\partial f}{\partial y} = x - \frac{y}{\sqrt{9-x^2-y^2}}$$

Notice that the partial derivatives are defined on $x^2+y^2<9$. Let us find the critical points:

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{array}\right\} \left.\begin{array}{l} x = y\sqrt{9-x^2-y^2} \\ y = x\sqrt{9-x^2-y^2} \end{array}\right\} \Rightarrow x = x(9-x^2-y^2)$$

$$\Rightarrow x(8-x^2-y^2)=0 \begin{cases} x=0 \Rightarrow y=0 \rightarrow (0,0) \\ 8-x^2-y^2=0 \begin{cases} (2,2) \\ (-2,-2) \end{cases} \end{cases}$$

2. <u>Boundary</u>: $x^2+y^2=9$

- Lagrange multipliers: Let $g(x,y) = x^2+y^2$

$$\left.\begin{array}{l} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 9 \end{array}\right\} \left.\begin{array}{l} \left\langle y - \frac{x}{\sqrt{9-x^2-y^2}}, x - \frac{y}{\sqrt{9-x^2-y^2}} \right\rangle = \lambda\langle 2x, 2y\rangle \\ x^2+y^2=9 \end{array}\right. \Rightarrow \text{No solution}$$

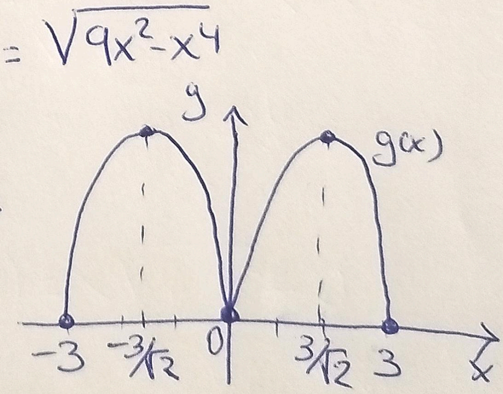
$\nabla f(x,y)$ is not defined

- Reducing to one variable:

$x^2+y^2=9 \Rightarrow y^2 = 9-x^2 \Rightarrow y(x) = \pm\sqrt{9-x^2}$

Consider $g(x) = f(x,y(x)) = x\sqrt{9-x^2} = \sqrt{9x^2-x^4}$

$$g'(x) = \frac{18x-4x^3}{2\sqrt{9x^2-x^4}} = \frac{9x-2x^3}{\sqrt{9x^2-x^4}} \underset{x\neq 0}{=} \frac{9-2x^2}{\sqrt{9-x^2}}$$



- Critical points:

$g'(x)=0 \Leftrightarrow x = \pm\frac{3}{\sqrt{2}} \Rightarrow y = \pm 3/\sqrt{2}$

$\rightarrow \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), \left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

- Singular points: $x=0 \Rightarrow y=\pm 3$
- Endpoints: $x=\pm 3 \Rightarrow y=0$

Remark: for $y(x) = -\sqrt{9-x^2}$ we obtain the same critical points.

3. Possible location for absolute extrema:

location	value f
$(0,0)$	3
$(2,2)$	5
$(-2,-2)$	5
$(0,3)$	0
$(0,-3)$	0
$(3,0)$	0
$(-3,0)$	0
$\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$	$\frac{9}{2}$
$\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$	$\frac{9}{2}$
$\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$	$-\frac{9}{2}$
$\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$	$-\frac{9}{2}$

$\rightarrow$ Maximum value $f(x,y) = 5$

$\rightarrow$ Minimum value $f(x,y) = -9/2$

with(*LinearAlgebra*) *with*(*VectorCalculus*) *with*(*Student*[*Calculus1*])*with*(*plots*) :

Exercise 4. Consider the two variables functions

$f := x^2 + y^2 - 9$

$$f := x^2 + y^2 - 9 \quad (1)$$

$g := 9\,x^2 - y^2 - 9$

$$g := 9\,x^2 - y^2 - 9 \quad (2)$$

We want to find the solution(s) of [f(x,y), g(x,y)] = [0, 0].

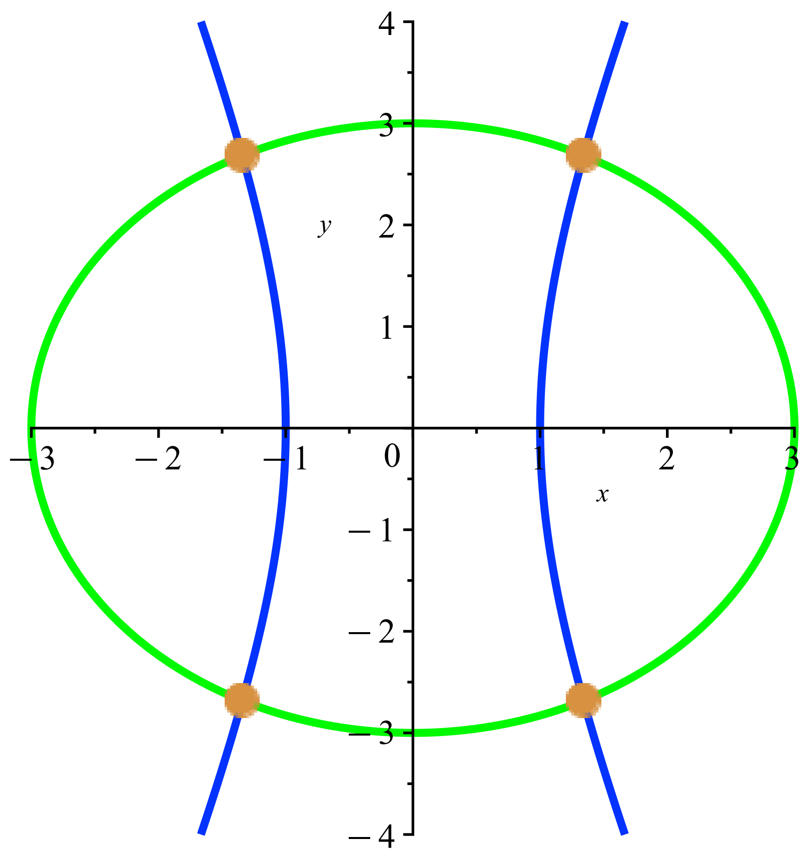
$xyinter := implicitplot([f = 0, g = 0], x = -4..4, y = -4..4, color = [green, blue], thickness = 3):$

The exact solutions can be easily computed and are the points

$\left(\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right), \left(-\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right), \left(-\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}\right)$ **and** $\left(\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}\right)$.

$$exactsol := pointplot\left(\left[\left[\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right], \left[-\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right], \left[-\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}\right], \left[\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}\right]\right], color = gold, symbolsize = 30, symbol = solidcircle\right):$$

$display(xyinter, exactsol)$



$a := evalf[15]\left(\frac{3}{\sqrt{5}}\right)$

$$a := 1.34164078649987 \tag{3}$$

$b := evalf[15]\left(\frac{6}{\sqrt{5}}\right)$

$$b := 2.68328157299975 \tag{4}$$

Approximation with Newton's method:

$F := Matrix(2, 1, [f, g])$

$$F := \begin{bmatrix} x^2 + y^2 - 9 \\ 9x^2 - y^2 - 9 \end{bmatrix} \tag{5}$$

$J := Jacobian([f, g], [x, y])$

$$J := \begin{bmatrix} 2x & 2y \\ 18x & -2y \end{bmatrix} \tag{6}$$

$Jinv := MatrixInverse(J)$

$$Jinv := \begin{bmatrix} \frac{1}{20x} & \frac{1}{20x} \\ \frac{9}{20y} & -\frac{1}{20y} \end{bmatrix} \tag{7}$$

$x[0] := 2; y[0] := 2$

$$x_0 := 2 \tag{8}$$

$$y_0 := 2 \tag{8}$$

for i **from** 0 **to** 4 **do** $A := Matrix(2, 1, [x[i], y[i]]) - subs(x = x[i], y = y[i], Jinv) \cdot subs(x = x[i], y = y[i], F) : x[i+1] := evalf(A[1, 1]); y[i+1] := evalf(A[2, 1]);$ **end**;

$$A := \begin{bmatrix} \frac{29}{20} \\ \frac{14}{5} \end{bmatrix}$$

$$x_1 := 1.450000000$$

$$y_1 := 2.800000000$$

$$A := \begin{bmatrix} 1.34568965517450 \\ 2.68571428573370 \end{bmatrix}$$

$$x_2 := 1.34568965517450$$

$$y_2 := 2.68571428573370$$

$$A := \begin{bmatrix} 1.34164687755418 \\ 2.68328267477205 \end{bmatrix}$$

$$x_3 := 1.34164687755418$$

$$y_3 := 2.68328267477205$$

$$A := \begin{bmatrix} 1.34164078651370 \\ 2.68328157299997 \end{bmatrix}$$

$$x_4 := 1.34164078651370$$

$$y_4 := 2.68328157299997$$

$$A := \begin{bmatrix} 1.34164078649987 \\ 2.68328157299975 \end{bmatrix}$$

$$x_5 := 1.34164078649987$$

$$y_5 := 2.68328157299975 \tag{9}$$