MEC-E8003 Beam, Plate and Shell Models; formulae

TENSORS

$$\vec{a}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}a_\alpha\\ a_\beta\\ a_\gamma\end{Bmatrix},\quad \vec{\vec{a}}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma}\\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma}\\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma}\end{bmatrix}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix},\ \text{etc.}$$

$$\vec{I}\cdot\vec{a}=\vec{a}\cdot\vec{I}=\vec{a}\quad\forall\vec{a},\quad \vec{I}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

$$\vec{\vec{I}}:\vec{\vec{a}}=\vec{\vec{a}}:\vec{\vec{I}}=\vec{\vec{a}}\quad\forall\vec{\vec{a}},\quad \vec{\vec{I}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}+\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}+\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}}\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}$$

$$\vec{\vec{a}}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma}\\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma}\\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma}\end{bmatrix}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\ \Leftrightarrow\ \vec{\vec{a}}_{\mathrm{c}}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma}\\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma}\\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma}\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}$$

$$\vec{\vec{a}}=-\vec{\vec{a}}_{\mathrm{c}}\ \Rightarrow\ \vec{\vec{a}}\cdot\vec{b}=\vec{a}\times\vec{b}\quad\forall\vec{b}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

$$\vec{\vec{a}}:(\nabla\vec{b})_{\mathrm{c}}=\nabla\cdot(\vec{\vec{a}}\cdot\vec{b})-(\nabla\cdot\vec{\vec{a}})\cdot\vec{b}$$

CURVILINEAR COORDINATES

$$\vec{r}(\alpha,\beta,\gamma)=x(\alpha,\beta,\gamma)\vec{i}+y(\alpha,\beta,\gamma)\vec{j}+z(\alpha,\beta,\gamma)\vec{k}$$

$$\begin{Bmatrix}\vec{h}_\alpha\\ \vec{h}_\beta\\ \vec{h}_\gamma\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial\vec{r}}{\partial\alpha}\\ \dfrac{\partial\vec{r}}{\partial\beta}\\ \dfrac{\partial\vec{r}}{\partial\gamma}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\begin{Bmatrix}\vec{h}_\alpha/h_\alpha\\ \vec{h}_\beta/h_\beta\\ \vec{h}_\gamma/h_\gamma\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},$$

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\quad \eta\in\{\alpha,\beta,\gamma\},\quad \nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix}\frac{\partial}{\partial\alpha}\\ \frac{\partial}{\partial\beta}\\ \frac{\partial}{\partial\gamma}\end{Bmatrix}$$

$$\nabla=\vec{e}_\alpha\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}+\vec{e}_\beta\frac{1}{h_\beta}\frac{\partial}{\partial\beta}+\vec{e}_\gamma\frac{1}{h_\gamma}\frac{\partial}{\partial\gamma}\quad \text{(orthonormal coordinate system)}$$

CYLINDRICAL COORDINATES

$$\vec{r}(r,\phi,z)=r(\cos\phi\vec{i}+\sin\phi\vec{j})+z\vec{k}$$

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}\mathrm{c}\phi & \mathrm{s}\phi & 0\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\\ 0\end{Bmatrix},\quad \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\frac{\partial}{\partial z}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=0$$

$$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}$$

SPHERICAL COORDINATES

$$\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})$$

$$\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{Bmatrix}\mathrm{c}\theta\vec{e}_\phi\\ -\mathrm{s}\theta\vec{e}_r-\mathrm{c}\theta\vec{e}_\theta\\ \mathrm{s}\theta\vec{e}_\phi\end{Bmatrix},\quad \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{Bmatrix}-\vec{e}_r\\ 0\\ \vec{e}_\theta\end{Bmatrix}$$

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=0,\quad \nabla=\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}+\vec{e}_r\frac{\partial}{\partial r}$$

BEAM COORDINATES

$$\vec{r}(s,n,b)=\vec{r}_0(s)+n\vec{e}_n+b\vec{e}_b$$

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\frac{\partial\vec{r}_0}{\partial s}\\ \frac{\partial\vec{e}_s}{\partial s}/|\frac{\partial\vec{e}_s}{\partial s}|\\ \vec{e}_s\times\vec{e}_n\end{Bmatrix},\quad \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\kappa\vec{e}_n\\ \tau\vec{e}_b-\kappa\vec{e}_s\\ -\tau\vec{e}_n\end{Bmatrix},\quad \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\frac{\partial}{\partial b}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=0$$

$$\nabla = \frac{\vec{e}_s}{1-n\kappa}[\frac{\partial}{\partial s} + \tau(b\frac{\partial}{\partial n} - n\frac{\partial}{\partial b})] + \vec{e}_n\frac{\partial}{\partial n} + \vec{e}_b\frac{\partial}{\partial b}$$

SHELL COORDINATES

$$\vec{r}(\alpha,\beta,n) = \vec{r}_0(\alpha,\beta) + n\vec{e}_n$$

$$\begin{Bmatrix}\vec{h}_\alpha \\ \vec{h}_\beta\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial\vec{r}_0}{\partial\alpha} \\ \dfrac{\partial\vec{r}_0}{\partial\beta}\end{Bmatrix},\ \begin{Bmatrix}\vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n\end{Bmatrix} = \begin{Bmatrix}\vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{e}_\alpha\times\vec{e}_\beta\end{Bmatrix} = [F]\begin{Bmatrix}\vec{i} \\ \vec{j} \\ \vec{k}\end{Bmatrix},\ \frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n\end{Bmatrix} = (\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n\end{Bmatrix}\quad \eta\in\{\alpha,\beta,n\},$$

$$\begin{Bmatrix}\dfrac{\partial\vec{r}}{\partial\alpha} \\ \dfrac{\partial\vec{r}}{\partial\beta} \\ \dfrac{\partial\vec{r}}{\partial n}\end{Bmatrix} = \begin{Bmatrix}\vec{h}_\alpha + n\dfrac{\partial\vec{e}_n}{\partial\alpha} \\ \vec{h}_\beta + n\dfrac{\partial\vec{e}_n}{\partial\beta} \\ \vec{e}_n\end{Bmatrix} = [H]\begin{Bmatrix}\vec{i} \\ \vec{j} \\ k\end{Bmatrix},\ \nabla = \begin{Bmatrix}\vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix}\dfrac{\partial}{\partial\alpha} \\ \dfrac{\partial}{\partial\beta} \\ \dfrac{\partial}{\partial n}\end{Bmatrix}$$

CYLINDRICAL SHELL COORDINATES

$$\vec{r}(z,\phi,n) = (R-n)(\cos\phi\vec{i} + \sin\phi\vec{j}) + z\vec{k}$$

$$\begin{Bmatrix}\vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n\end{Bmatrix} = \begin{bmatrix}0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0\end{bmatrix}\begin{Bmatrix}\vec{i} \\ \vec{j} \\ \vec{k}\end{Bmatrix},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n\end{Bmatrix} = \begin{Bmatrix}0 \\ \vec{e}_n \\ -\vec{e}_\phi\end{Bmatrix},\ \frac{\partial}{\partial z}\begin{Bmatrix}\vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n\end{Bmatrix} = \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n\end{Bmatrix} = 0$$

$$\nabla = \vec{e}_z\frac{\partial}{\partial z} + (\frac{R}{R-n})\frac{1}{R}\vec{e}_\phi\frac{\partial}{\partial\phi} + \vec{e}_n\frac{\partial}{\partial n}$$

SPHERICAL SHELL COORDINATES

$$\vec{r}(\phi,\theta,n) = (R-n)(\sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k})$$

$$\begin{Bmatrix}\vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n\end{Bmatrix} = \begin{bmatrix}-\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta\end{bmatrix}\begin{Bmatrix}\vec{i} \\ \vec{j} \\ \vec{k}\end{Bmatrix},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n\end{Bmatrix} = \begin{Bmatrix}\sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta \\ \cos\theta\vec{e}_\phi \\ -\sin\theta\vec{e}_\phi\end{Bmatrix}$$

$$\frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n\end{Bmatrix} = \begin{Bmatrix}0 \\ \vec{e}_n \\ -\vec{e}_\theta\end{Bmatrix},\ \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n\end{Bmatrix} = 0,\ \nabla = \frac{R}{R-n}\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{R}{R-n}\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta} + \vec{e}_n\frac{\partial}{\partial n}$$

CIRCULAR PLATE COORDINATES

$$\vec{r}(r,\phi,n) = r(\cos\phi\vec{i} + \sin\phi\vec{j}) + n\vec{k}$$

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix}, \quad \frac{\partial}{\partial r}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \frac{\partial}{\partial n}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = 0$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_n \frac{\partial}{\partial n}$$

LINEAR ELASTICITY

$$\vec{\sigma} = \vec{\vec{E}} : \vec{\varepsilon} = \vec{\vec{E}} : \nabla\vec{u}, \quad \vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_c]$$

Elastic material: $\vec{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}$

Isotropic: $[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}$

Plane stress: $[E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Beam: $[E] = \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}$

Plate: $[E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix},$

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}, \quad [E]_\sigma^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}, \quad D = \frac{t^3 E}{12(1-\nu^2)}$$

PRINCIPLE OF VIRTUAL WORK

$$\delta W = \delta W^{\mathrm{ext}} + \delta W^{\mathrm{int}} = 0 \quad \forall \delta\vec{u} \in U \quad \text{(a function set)}$$

$$\delta W = -\int_V (\vec{\sigma} : \delta\vec{\varepsilon}_c) dV + \int_V (\vec{f} \cdot \delta\vec{u}) dV + \int_A (\vec{t} \cdot \delta\vec{u}) dA$$

BEAM EQUATIONS

$$\begin{Bmatrix} \dfrac{d\vec{F}}{ds} + \vec{b} \\ \dfrac{d\vec{M}}{ds} + \vec{e}_s \times \vec{F} + \vec{c} \end{Bmatrix} = 0\,,\ \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho} \times \vec{f} \end{Bmatrix} JdA\,,\ \begin{Bmatrix} \vec{F} \\ \vec{\underline{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho} \times \vec{t} \end{Bmatrix} JdA, \text{ where } J = 1 + n\kappa$$

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix} dA = \int \begin{bmatrix} \vec{E} & -\vec{E} \times \vec{\rho} \\ \vec{\rho} \times \vec{E} & -\vec{\rho} \times \vec{E} \times \vec{\rho} \end{bmatrix} dA \cdot \begin{Bmatrix} \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \\ \dfrac{d\vec{\theta}_0}{ds} \end{Bmatrix}, \text{ where } \vec{E} = \vec{e}_s \cdot \vec{\vec{E}} \cdot \vec{e}_s$$

TIMOSHENKO BEAM (x, y, z)

$$\vec{u}_0 = u\vec{i} + v\vec{j} + w\vec{k}\,,\ \vec{\theta}_0 = \phi\vec{i} + \theta\vec{j} + \psi\vec{k}\,,\ \vec{F} = N\vec{i} + Q_y\vec{j} + Q_z\vec{k}\,,\ \vec{M} = T\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0\,,\ \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} - ES_z\dfrac{d\psi}{dx} + ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx} - \psi) - GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx} + \theta) + GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0\,,\ \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(\dfrac{dv}{dx} - \psi) + GS_z(\dfrac{dw}{dx} + \theta) + GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx} - EI_{zy}\dfrac{d\psi}{dx} + EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx} + EI_{zz}\dfrac{d\psi}{dx} - EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

TIMOSHENKO BEAM (s, n, b)

$$\vec{u} = u\vec{e}_s + v\vec{e}_n + w\vec{e}_b\,,\ \vec{\theta} = \phi\vec{e}_s + \theta\vec{e}_n + \psi\vec{e}_b\,, \vec{F} = N\vec{e}_s + Q_n\vec{e}_n + Q_b\vec{e}_b\,,\ \vec{M} = T\vec{e}_s + M_n\vec{e}_n + M_b\vec{e}_b$$

$$\begin{Bmatrix} \dfrac{dN}{ds} - Q_n\kappa + b_s \\ \dfrac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \dfrac{dQ_b}{ds} + Q_n\tau + b_b \end{Bmatrix} = 0\,,\ \begin{Bmatrix} \dfrac{dT}{ds} - M_n\kappa + c_s \\ \dfrac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \dfrac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\frac{du}{ds} - v\kappa) + ES_n(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - ES_b(\frac{d\psi}{ds} + \theta\tau) \\ GA(\frac{dv}{ds} + u\kappa - w\tau - \psi) - GS_n(\frac{d\phi}{ds} - \theta\kappa) \\ GA(\frac{dw}{ds} + v\tau + \theta) + GS_b(\frac{d\phi}{ds} - \theta\kappa) \end{Bmatrix}$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(\frac{dw}{ds} + v\tau + \theta) + GI_{rr}(\frac{d\phi}{ds} - \theta\kappa) - GS_n(\frac{dv}{ds} + u\kappa - w\tau - \psi) \\ ES_n(\frac{du}{ds} - v\kappa) + EI_{nn}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - EI_{bn}(\frac{d\psi}{ds} + \theta\tau) \\ -ES_b(\frac{du}{ds} - v\kappa) - EI_{nb}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) + EI_{bb}(\frac{d\psi}{ds} + \theta\tau) \end{Bmatrix}$$

PLATE EQUATIONS

$\nabla \cdot \vec{\vec{F}} + \vec{b} = 0$, $\nabla \cdot \vec{\vec{M}} - \vec{e}_n \cdot \vec{\vec{F}} + \vec{c} = 0$,

$$\begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} = \int \vec{\vec{\sigma}} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} \vec{\vec{\vec{\vec{E}}}} dn : \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}, \quad \vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$$

REISSNER-MINDLIN PLATE (x, y, n)

$$\vec{\vec{F}} = \int \vec{\vec{\sigma}} dn = \vec{i}\vec{i} N_{xx} + (\vec{i}\vec{j} + \vec{j}\vec{i}) N_{xy} + \vec{j}\vec{j} N_{yy} + (\vec{e}_n \vec{i} + \vec{i}\vec{e}_n) Q_x + (\vec{e}_n \vec{j} + \vec{j}\vec{e}_n) Q_y$$

$$\vec{\vec{M}} = \int \vec{\vec{\sigma}} n dn = \vec{i}\vec{i} M_{xx} + (\vec{i}\vec{j} + \vec{j}\vec{i}) M_{xy} + \vec{j}\vec{j} M_{yy}$$

$$\begin{Bmatrix} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + b_x \\ \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0 , \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_\sigma \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + b_n \\ \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \\ \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0 , \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \frac{\partial \theta}{\partial x} \\ -\frac{\partial \phi}{\partial y} \\ \frac{\partial \theta}{\partial y} - \frac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \frac{\partial w}{\partial x} + \theta \\ \frac{\partial w}{\partial y} - \phi \end{Bmatrix}$$

REISSNER-MINDLIN PLATE (r, ϕ, n)

$$\left\{\begin{matrix} \frac{1}{r}[\frac{\partial(rN_{rr})}{\partial r}+\frac{\partial N_{r\phi}}{\partial\phi}-N_{\phi\phi}]+b_r \\ \frac{1}{r}[\frac{1}{r}\frac{\partial(r^2N_{r\phi})}{\partial r}+\frac{\partial N_{\phi\phi}}{\partial\phi}]+b_\phi \end{matrix}\right\}=0\,,\quad \left\{\begin{matrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{matrix}\right\}=t\left[E\right]_\sigma\left\{\begin{matrix} \frac{\partial u_r}{\partial r} \\ \frac{1}{r}(u_r+\frac{\partial u_\phi}{\partial\phi}) \\ \frac{1}{r}\frac{\partial u_r}{\partial\phi}+r\frac{\partial}{\partial r}(\frac{u_\phi}{r}) \end{matrix}\right\}$$

$$\left\{\begin{matrix} \frac{1}{r}[\frac{\partial(rQ_r)}{\partial r}+\frac{\partial Q_\phi}{\partial\phi}]+b_n \\ \frac{1}{r}[\frac{\partial(rM_{rr})}{\partial r}+\frac{\partial M_{r\phi}}{\partial\phi}-M_{\phi\phi}]-Q_r \\ \frac{1}{r}[\frac{\partial(rM_{r\phi})}{\partial r}+\frac{\partial M_{\phi\phi}}{\partial\phi}+M_{r\phi}]-Q_\phi \end{matrix}\right\}=0\,,\quad \left\{\begin{matrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{matrix}\right\}=\frac{t^3}{12}\left[E\right]_\sigma\left\{\begin{matrix} \frac{\partial\theta_\phi}{\partial r} \\ \frac{1}{r}(\theta_\phi-\frac{\partial\theta_r}{\partial\phi}) \\ \frac{1}{r}(\frac{\partial\theta_\phi}{\partial\phi}+\theta_r)-\frac{\partial\theta_r}{\partial r} \end{matrix}\right\}$$

$$\left\{\begin{matrix} Q_r \\ Q_\phi \end{matrix}\right\}=Gt\left\{\begin{matrix} \frac{\partial w}{\partial r}+\theta_\phi \\ \frac{1}{r}\frac{\partial w}{\partial\phi}-\theta_r \end{matrix}\right\}$$

KIRCHHOFF PLATE BENDING (r,ϕ,n)

$$\nabla_0^2\nabla_0^2 w-\frac{b_n}{D}=0\,,\quad w(r)=\frac{b_n}{D}\frac{r^4}{64}+a+br^2+cr^2(1-\log r)+d\log r$$

MEMBRANE EQUATIONS IN CYLINDRICAL GEOMETRY (z,ϕ,n)

$$\left\{\begin{matrix} \frac{1}{R}\frac{\partial N_{z\phi}}{\partial\phi}+\frac{\partial N_{zz}}{\partial z}+b_z \\ \frac{\partial N_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi}+b_\phi \\ \frac{1}{R}N_{\phi\phi}+b_n \end{matrix}\right\}=0\,,\quad \left\{\begin{matrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{matrix}\right\}=t\left[E\right]_\sigma\left\{\begin{matrix} \frac{\partial u_z}{\partial z} \\ \frac{1}{R}(\frac{\partial u_\phi}{\partial\phi}-u_n) \\ \frac{1}{R}\frac{\partial u_z}{\partial\phi}+\frac{\partial u_\phi}{\partial z} \end{matrix}\right\}$$

MEMBRANE EQUATIONS IN SPHERICAL GEOMETRY (ϕ,θ,n)

$$\left\{\begin{matrix} \frac{1}{R}(\csc\theta\frac{\partial N_{\phi\phi}}{\partial\phi}+\frac{\partial N_{\phi\theta}}{\partial\theta}+2\cot\theta N_{\phi\theta})+b_\phi \\ \frac{1}{R}[\csc\theta\frac{\partial N_{\phi\theta}}{\partial\phi}+\frac{\partial N_{\theta\theta}}{\partial\theta}+\cot\theta(N_{\theta\theta}-N_{\phi\phi})]+b_\theta \\ \frac{1}{R}(N_{\phi\phi}+N_{\theta\theta})+b_n \end{matrix}\right\}=0$$

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix} = t\left[E\right]_{\sigma} \begin{Bmatrix} \frac{1}{R}[\csc\theta(\cos\theta u_{\theta} + \frac{\partial u_{\phi}}{\partial\phi}) - u_n] \\ \frac{1}{R}(\csc\theta\sin\theta\frac{\partial u_{\theta}}{\partial\theta} - u_n) \\ \frac{1}{R}(\csc\theta\frac{\partial u_{\theta}}{\partial\phi} - \cot\theta u_{\phi} + \frac{\partial u_{\phi}}{\partial\theta}) \end{Bmatrix} \quad (\csc\theta = \frac{1}{\sin\theta})$$

SHELL EQUATIONS IN CYLINDRICAL GEOMETRY (z,ϕ,n)

$$\begin{Bmatrix} \frac{1}{R}\frac{\partial N_{\phi z}}{\partial\phi} + \frac{\partial N_{zz}}{\partial z} + b_z \\ \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi} - \frac{1}{R}Q_{\phi} + b_{\phi} \end{Bmatrix} = 0\,, \quad \begin{Bmatrix} \frac{1}{R}\frac{\partial Q_{\phi}}{\partial\phi} + \frac{\partial Q_z}{\partial z} + \frac{1}{R}N_{\phi\phi} + b_n \\ \frac{\partial M_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial M_{\phi\phi}}{\partial\phi} - \frac{1}{R}M_{\phi n} - Q_{\phi} + c_{\phi} \\ \frac{\partial M_{zz}}{\partial z} + \frac{1}{R}\frac{\partial M_{\phi z}}{\partial\phi} - Q_z + c_z \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \\ N_{\phi z} \end{Bmatrix} = \begin{Bmatrix} \frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z} + \nu\frac{1}{R}(\frac{\partial u_{\phi}}{\partial\phi} - u_n)] - D\frac{1}{R}\frac{\partial\theta_{\phi}}{\partial z} \\ \frac{tE}{1-\nu^2}[\frac{1}{R}(\frac{\partial u_{\phi}}{\partial\phi} - u_n) + \nu\frac{\partial u_z}{\partial z}] - D\frac{1}{R^2}\frac{\partial\theta_z}{\partial\phi} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial\phi} + \frac{\partial u_{\phi}}{\partial z}) + \frac{1}{2}(1-\nu)D\frac{1}{R}\frac{\partial\theta_z}{\partial z} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial\phi} + \frac{\partial u_{\phi}}{\partial z}) + \frac{1}{2}(1-\nu)D\frac{1}{R^2}\frac{\partial\theta_{\phi}}{\partial\phi} \end{Bmatrix}, \quad \begin{Bmatrix} Q_z \\ Q_{\phi} \end{Bmatrix} = tG\begin{Bmatrix} \frac{\partial u_n}{\partial z} + \theta_{\phi} \\ \frac{1}{R}(\frac{\partial u_n}{\partial\phi} + u_{\phi}) - \theta_z \end{Bmatrix}$$

$$\begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \end{Bmatrix} = D\begin{Bmatrix} \frac{\partial\theta_{\phi}}{\partial z} - \nu\frac{1}{R}\frac{\partial\theta_z}{\partial\phi} - \frac{1}{R}\frac{\partial u_z}{\partial z} \\ \nu\frac{\partial\theta_{\phi}}{\partial z} - \frac{1}{R}\frac{\partial\theta_z}{\partial\phi} + \frac{1}{R^2}(\frac{\partial u_{\phi}}{\partial\phi} - u_n) \\ \frac{1}{2}(1-\nu)[(\frac{1}{R}\frac{\partial\theta_{\phi}}{\partial\phi} - \frac{\partial\theta_z}{\partial z}) - \frac{1}{R}\frac{\partial u_{\phi}}{\partial z}] \\ \frac{1}{2}(1-\nu)[(\frac{1}{R}\frac{\partial\theta_{\phi}}{\partial\phi} - \frac{\partial\theta_z}{\partial z}) + \frac{1}{R^2}\frac{\partial u_z}{\partial\phi}] \end{Bmatrix}, \quad M_{\phi n} = \frac{1}{2}(1-\nu)D\frac{1}{R}[\frac{1}{R}(\frac{\partial u_n}{\partial\phi} + u_{\phi}) - \theta_z]$$

SHELL EQUATIONS IN SPHERICAL GEOMETRY (ϕ,θ,n)

$$\begin{Bmatrix} \frac{1}{R}(\frac{\partial}{\partial\theta}N_{\phi\theta} + \csc\theta\frac{\partial}{\partial\phi}N_{\phi\phi} + 2\cot\theta N_{\phi\theta} - Q_{\phi}) + b_{\phi} \\ \frac{1}{R}(\frac{\partial}{\partial\theta}N_{\theta\theta} + \csc\theta\frac{\partial}{\partial\phi}N_{\phi\theta} + \cot\theta N_{\theta\theta} - \cot\theta N_{\phi\phi} - Q_{\theta}) + b_{\theta} \end{Bmatrix} = 0\,,$$

$$\left\{\begin{array}{c}\frac{1}{R}(\frac{\partial}{\partial\theta}Q_\theta+\csc\theta\frac{\partial}{\partial\phi}Q_\phi+\cot\theta Q_\theta+N_{\theta\theta}+N_{\phi\phi})+b_n\\ \frac{1}{R}(\frac{\partial}{\partial\theta}M_{\phi\theta}+\csc\theta\frac{\partial}{\partial\phi}M_{\phi\phi}+2\cot\theta M_{\phi\theta})-Q_\phi+c_\phi\\ \frac{1}{R}(\frac{\partial}{\partial\theta}M_{\theta\theta}+\csc\theta\frac{\partial}{\partial\phi}M_{\phi\theta}+\cot\theta M_{\theta\theta}-\cot\theta M_{\phi\phi})-Q_\theta+c_\theta\end{array}\right\}=0,$$

$$\left\{\begin{array}{c}N_{\phi\phi}\\ N_{\theta\theta}\\ N_{\phi\theta}\end{array}\right\}=\frac{Et}{1-\nu^2}\frac{1}{R}\left\{\begin{array}{c}(u_\theta\cot\theta+\frac{\partial u_\phi}{\partial\phi}\csc\theta-u_n)+\nu(\frac{\partial u_\theta}{\partial\theta}-u_n)\\ \nu(u_\theta\cot\theta+\frac{\partial u_\phi}{\partial\phi}\csc\theta-u_n)+(\frac{\partial u_\theta}{\partial\theta}-u_n)\\ \frac{1-\nu}{2}(-u_\phi\cot\theta+\frac{\partial u_\theta}{\partial\phi}\csc\theta+\frac{\partial u_\phi}{\partial\theta})\end{array}\right\},$$

$$\left\{\begin{array}{c}M_{\phi\phi}\\ M_{\theta\theta}\\ M_{\phi\theta}\end{array}\right\}=D\frac{1}{R}\left\{\begin{array}{c}-\theta_\phi\cot\theta+\frac{\partial\theta_\theta}{\partial\phi}\csc\theta-\nu\frac{\partial\theta_\phi}{\partial\theta}\\ \nu(-\theta_\phi\cot\theta+\frac{\partial\theta_\theta}{\partial\phi}\csc\theta)-\frac{\partial\theta_\phi}{\partial\theta}\\ \frac{1-\nu}{2}(\frac{\partial\theta_\theta}{\partial\theta}-\theta_\theta\cot\theta-\frac{\partial\theta_\phi}{\partial\phi}\csc\theta)\end{array}\right\},\quad \left\{\begin{array}{c}Q_\phi\\ Q_\theta\end{array}\right\}=tG\left\{\begin{array}{c}\theta_\theta+\frac{1}{R}(u_\phi+\frac{\partial u_n}{\partial\phi}\csc\theta)\\ -\theta_\phi+\frac{1}{R}(u_\theta+\frac{\partial u_n}{\partial\theta})\end{array}\right\}$$