The Nyquist Stability Criterion

Elham Abolfazli
Office: room 1305a, Otakaari 5
Email: elham.abolfazli@aalto.fi

In the previous lecture...

You:

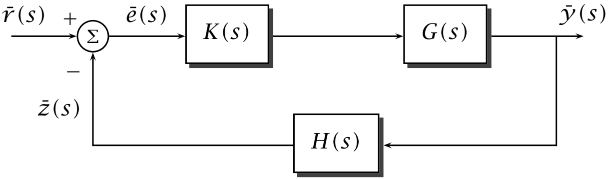
- Understood the powerful concept of frequency response and its role in control system design.
- Knew how to sketch a Bode plot and also how to obtain a computer-generated Bode plot.
- Became familiar with log magnitude and phase diagrams.

Learning outcomes...

...the student will:

- Learn how to plot the *open-loop* frequency response of the return ratio $L(s)$
- Use the Nyquist diagram to ascertain stability of the *closed-loop* system
- Infer more detailed information about the behavior of the closed-loop system

The Return Ratio



- The Return Ratio of a loop is defined as -1 times the product of all the terms around the loop. In this case,

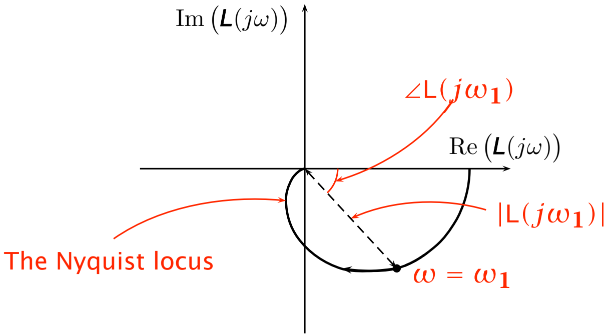
$$L(s) = (-1) \times (-H(s))K(s)G(s) = H(s)K(s)G(s)$$

- Feedback control systems are often tested in this configuration as a final check before closing the loop. Note that

$$\bar{e}(s) = \frac{1}{1 + L(s)}\bar{r}(s)$$

$$\bar{y}(s) = \frac{K(s)G(s)}{1 + L(s)}\bar{r}(s)$$

The Nyquist diagram



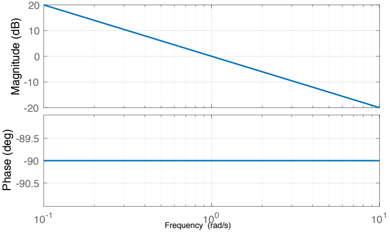
Example: Consider the integrator

$$G(s) = 1/s$$

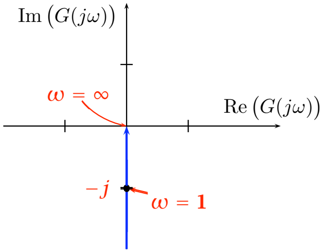
Then,

$$G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega} \Rightarrow \begin{cases} |G(j\omega)| = 1/\omega \\ \angle G(j\omega) = -90^o \end{cases}$$

Recall the **bode plot**:



The **Nyquist plot** is therefore:



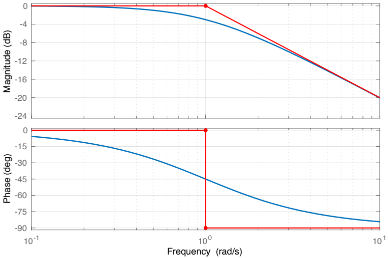
Example: Consider the system with first-order lag

$$G(s) = \frac{1}{1+sT}$$

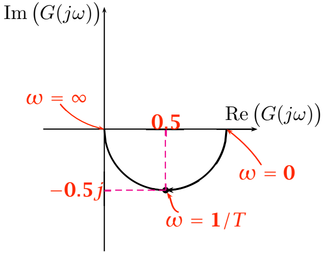
Then,

$$G(j\omega) = \frac{1}{1+j\omega T} \Rightarrow \begin{cases} |G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T^2}} \\ \angle G(j\omega) = -\arctan(\omega T) \end{cases}$$

Recall the **bode plot** ($T=1$):



The **Nyquist plot** is therefore:



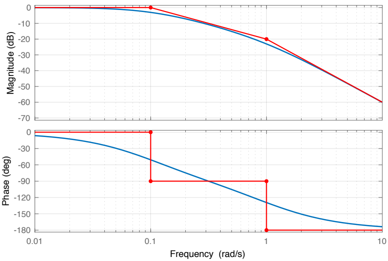
Example: Consider the system with second-order lag

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$$

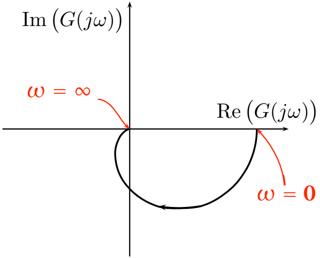
Then,

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} \Rightarrow \begin{cases} |G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}} \\ \angle G(j\omega) = -\arctan(\omega T_1) - \arctan(\omega T_2) \end{cases}$$

Recall the **bode plot** ($T_1 = 1$, $T_2 = 10$):



The **Nyquist plot** is therefore:



Example: Consider the system with time delay, lag and integrator

$$G(s) = \frac{e^{-sT_1}}{s(1+sT_2)}$$

Then,

$$G(j\omega) = \frac{e^{-j\omega T_1}}{j\omega(1+j\omega T_2)} \Rightarrow \begin{cases} |G(j\omega)| = \underbrace{|e^{-j\omega T_1}|}_{=1} \times \frac{1}{|j\omega|} \times \frac{1}{|1+j\omega T_2|} \\ \angle G(j\omega) = \underbrace{\angle e^{-j\omega T_1}}_{-\omega T_1} - \underbrace{\angle(j\omega)}_{90^o} - \angle(1+j\omega T_2) \end{cases}$$

- Clearly, as $\omega \to 0$, the $|G(j\omega)| \to \infty$. But this is not enough information to sketch the Nyquist diagram.
- **How does $|G(j\omega)| \to \infty$?**
  - To answer this, we use Taylor series expansion around $\omega = 0$

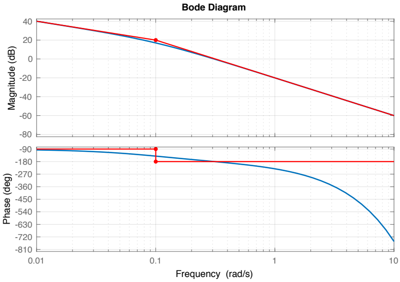
$$e^{-j\omega T_1} = \sum_{n=0}^{\infty} \frac{(-j\omega T_1)^n}{n!} = 1 + \frac{(-j\omega T_1)}{1!} + \frac{(-j\omega T_1)^2}{2!} + \ldots \approx 1 - j\omega T_1$$

$$\frac{1}{1+j\omega T_2} = 1 + (-j\omega T_2) + (-j\omega T_2)^2 + (-j\omega T_2)^3 + \ldots \approx 1 - j\omega T_2$$

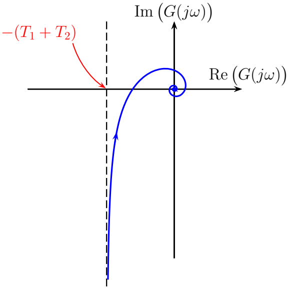
- **How does $|G(j\omega)| \to \infty$? (continued...)**
  - Therefore,

$$G(j\omega) \to \frac{(1 - j\omega T_1)(1 - j\omega T_2)}{j\omega} = \frac{1}{j\omega} - (T_1 + T_2) + j\omega T_1 T_2 \to 0$$

The **bode plot** ($T_1 = 1$, $T_2 = 10$):



The **Nyquist plot** is therefore:



Sketching Nyquist diagrams

- Unlike the Bode diagram, there are no detailed rules for sketching Nyquist diagrams
  - Suffices to determine the asymptotic behavior as $\omega \to 0$ and $\omega \to \infty$ (using the techniques we have seen in the examples) and then calculate a few points in between.
  - If $G(j0)$ is a finite and non-zero, then the Nyquist diagram will always start off by leaving the real axis at right angles to it. This is due to (Taylor series expansion):

$$G(j\varepsilon) = G(j0) + j\varepsilon G'(j0) - \varepsilon^2 G''(j0) - \ldots \approx G(j0) + j\varepsilon G'(j0)$$

  - If $G(j0)$ is infinite, due to the presence of integrators, then we must explicitly find the first two terms of the Taylor series expansion of $G(j\omega)$ about $\omega = 0$, as in the example with a time delay, a lag and an integrator.

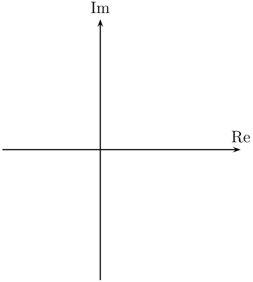
Feedback Stability

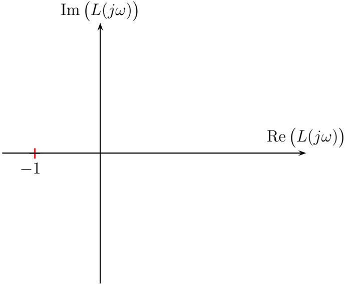
Definition (Asymptotic stability of a feedback system)

We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(s)/(1 + L(s))$ is asymptotically stable.

Closed-loop poles $\equiv$ poles of $\frac{L(s)}{1+L(s)}$ $\equiv$ roots of $1 + L(s) = 0$

- This corresponds to all the roots of $1 + C(s)P(s) = 0$ lying in the LHP.





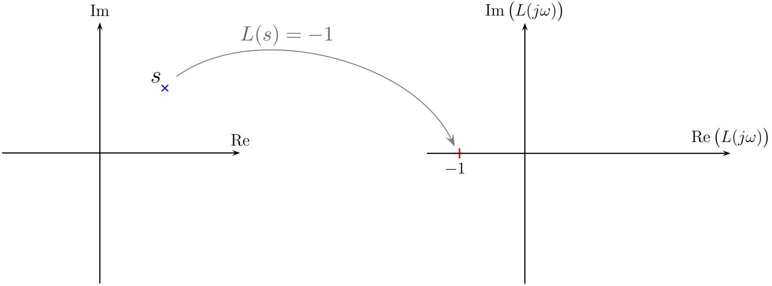
Feedback Stability

Definition (Asymptotic stability of a feedback system)

We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(s)/(1 + L(s))$ is asymptotically stable.

Closed-loop poles $\equiv$ poles of $\frac{L(s)}{1+L(s)}$ $\equiv$ roots of $1 + L(s) = 0$

- This corresponds to all the roots of $1 + C(s)P(s) = 0$ lying in the LHP.



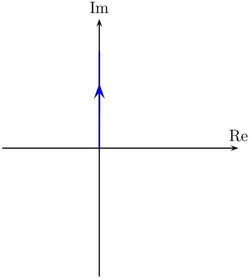
Feedback Stability

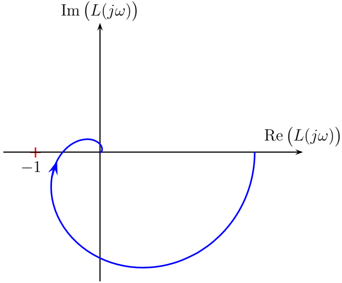
Definition (Asymptotic stability of a feedback system)

We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(s)/(1+L(s))$ is asymptotically stable.

Closed-loop poles $\equiv$ poles of $\frac{L(s)}{1+L(s)}$ $\equiv$ roots of $1+L(s)=0$

- This corresponds to all the roots of $1+C(s)P(s)=0$ lying in the LHP.





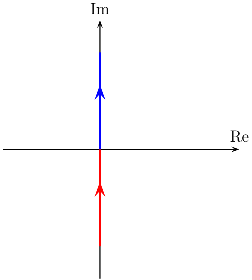
Feedback Stability

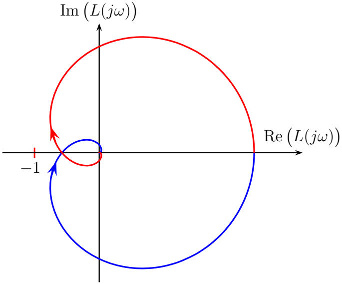
Definition (Asymptotic stability of a feedback system)

We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(s)/(1+L(s))$ is asymptotically stable.

Closed-loop poles $\equiv$ poles of $\frac{L(s)}{1+L(s)}$ $\equiv$ roots of $1+L(s)=0$

- This corresponds to all the roots of $1+C(s)P(s)=0$ lying in the LHP.





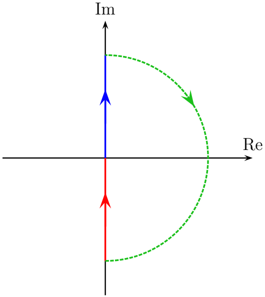
Feedback Stability

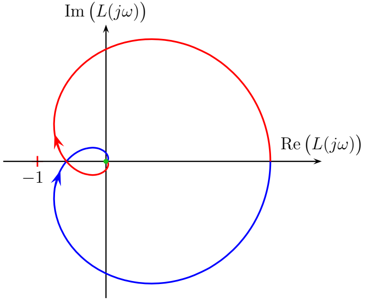
Definition (Asymptotic stability of a feedback system)

We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(s)/(1+L(s))$ is asymptotically stable.

Closed-loop poles $\equiv$ poles of $\frac{L(s)}{1+L(s)}$ $\equiv$ roots of $1+L(s)=0$

- This corresponds to all the roots of $1+C(s)P(s)=0$ lying in the LHP.





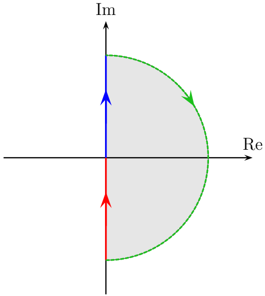
Feedback Stability

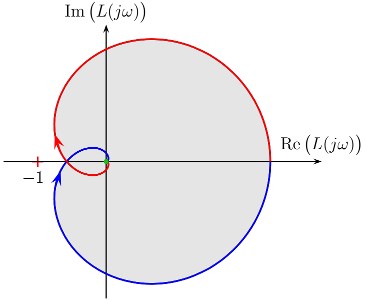
Definition (Asymptotic stability of a feedback system)

We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(s)/(1+L(s))$ is asymptotically stable.

Closed-loop poles $\equiv$ poles of $\frac{L(s)}{1+L(s)}$ $\equiv$ roots of $1+L(s)=0$

- This corresponds to all the roots of $1+C(s)P(s)=0$ lying in the LHP.



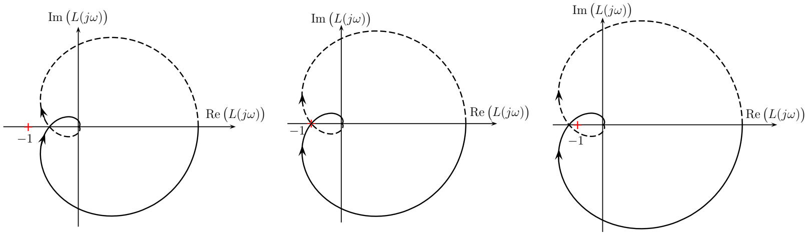


Nyquist's Stability Theorem

Simplified Nyquist's Stability Theorem

If a feedback system has a stable return ratio $L(s)$, then the feedback system is stable if and only if the closed contour given by $\Omega = \{L(j\omega) : -\infty < \omega < \infty\} \subset \mathbb{C}$ has no *net* encirclements of the point "-1".

- If $L(s)$ is stable (either marginally or asymptotically), then

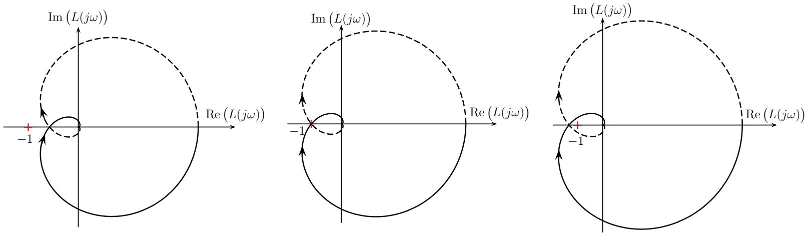


Nyquist's Stability Theorem

Simplified Nyquist's Stability Theorem

If a feedback system has a stable return ratio $L(s)$, then the feedback system is stable if and only if the closed contour given by $\Omega = \{L(j\omega) : -\infty < \omega < \infty\} \subset \mathbb{C}$ has no *net* encirclements of the point "-1".

- If $L(s)$ is stable (either marginally or asymptotically), then



$\Rightarrow \dfrac{L(s)}{1+L(s)}$ **asymptotically stable**

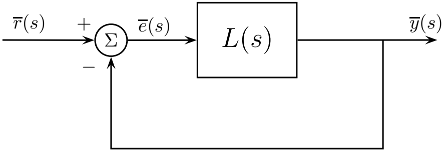
$\Rightarrow \dfrac{L(s)}{1+L(s)}$ **marginally stable**

$\Rightarrow \dfrac{L(s)}{1+L(s)}$ **unstable**

But what is the intuition behind it?

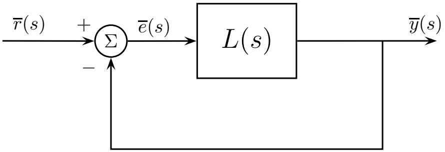
But what is the intuition behind it?

- Negative feedback is used to reduce the size of the error $e(t)$. If $y(t)$ is greater than $r(t)$, then $e(t)$ is negative.



But what is the intuition behind it?

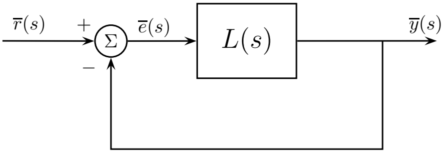
- Negative feedback is used to reduce the size of the error $e(t)$. If $y(t)$ is greater than $r(t)$, then $e(t)$ is negative.



- The phase lag from the input to the output $(-\angle L(j\omega))$ tends to increase with frequency, reaching $180°$. When this happens, the negative feedback is turned into positive feedback!

But what is the intuition behind it?

- Negative feedback is used to reduce the size of the error $e(t)$. If $y(t)$ is greater than $r(t)$, then $e(t)$ is negative.



- The phase lag from the input to the output $(-\angle L(j\omega))$ tends to increase with frequency, reaching 180°. When this happens, the negative feedback is turned into positive feedback!

- If the gain $|L(j\omega)|$ has not decreased to less than 1 by this frequency then instability of the closed-loop system will result.

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

The closed-loop poles are the roots of

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

The closed-loop poles are the roots of

$$1 + \frac{k}{s^3 + s^2 + 2s + 1} = 0 \quad \Rightarrow \quad s^3 + s^2 + 2s + 1 + k = 0$$

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

The closed-loop poles are the roots of

$$1 + \frac{k}{s^3 + s^2 + 2s + 1} = 0 \quad \Rightarrow \quad s^3 + s^2 + 2s + 1 + k = 0$$

The open-loop frequency response $L(j\omega)$ is given by

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

The closed-loop poles are the roots of

$$1 + \frac{k}{s^3 + s^2 + 2s + 1} = 0 \quad \Rightarrow \quad s^3 + s^2 + 2s + 1 + k = 0$$

The open-loop frequency response $L(j\omega)$ is given by

$$L(j\omega) = \frac{k}{j(-\omega^3 + 2\omega) + (-\omega^2 + 1)}$$

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

The closed-loop poles are the roots of

$$1 + \frac{k}{s^3 + s^2 + 2s + 1} = 0 \quad \Rightarrow \quad s^3 + s^2 + 2s + 1 + k = 0$$

The open-loop frequency response $L(j\omega)$ is given by

$$L(j\omega) = \frac{k}{j(-\omega^3 + 2\omega) + (-\omega^2 + 1)}$$

When the Nyquist diagram crosses the real axis, $L(j\omega_c)$ is purely real. In this case, it is real for $\omega_c = \sqrt{2}$, i.e.,

Example:

Let $P(s) = \dfrac{1}{s^3 + s^2 + 2s + 1}$ and $C(s) = k$, $k > 0$. Therefore, $L(s)$ is given by

$$L(s) = \frac{k}{s^3 + s^2 + 2s + 1}$$

The closed-loop poles are the roots of

$$1 + \frac{k}{s^3 + s^2 + 2s + 1} = 0 \quad \Rightarrow \quad s^3 + s^2 + 2s + 1 + k = 0$$

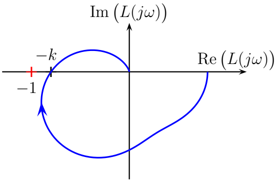
The open-loop frequency response $L(j\omega)$ is given by

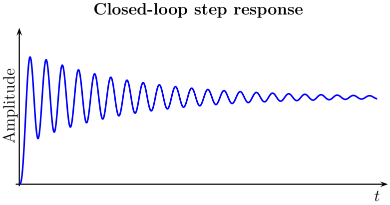
$$L(j\omega) = \frac{k}{j(-\omega^3 + 2\omega) + (-\omega^2 + 1)}$$

When the Nyquist diagram crosses the real axis, $L(j\omega_c)$ is purely real. In this case, it is real for $\omega_c = \sqrt{2}$, i.e.,

$$L(j\sqrt{2}) = \frac{k}{j(-(\sqrt{2})^3 + 2\sqrt{2}) + (-(\sqrt{2})^2 + 1)} = -k$$

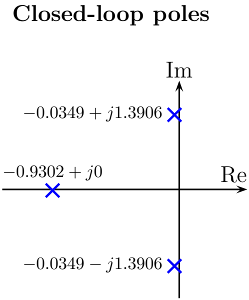
$k = 0.8$:



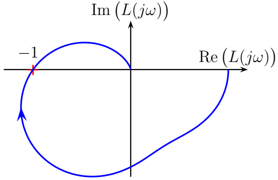


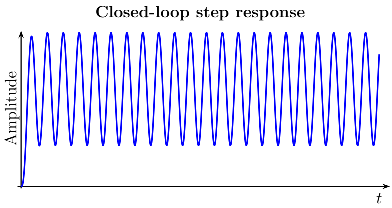
- Closed-loop system is *asymptotically* stable.
- Then, the oscillations decay.
- Closed-loop poles are the roots of $s^3 + s^2 + 2s + 1.8 = 0$, i.e.,

$$s = \{-0.9302, -0.0349 \pm j1.3906\}$$

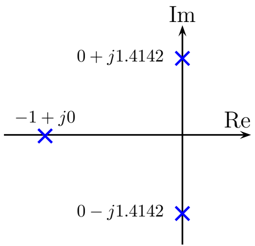


$k = 1$:





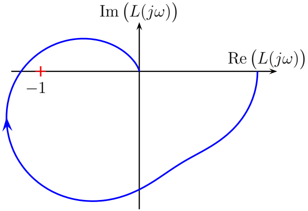
Closed-loop poles

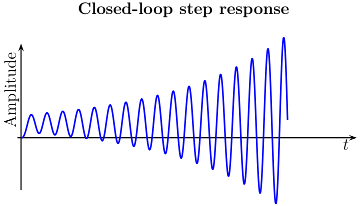


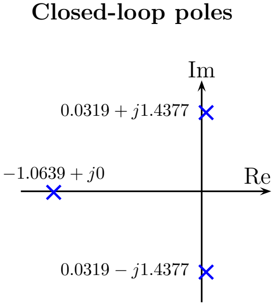
- Closed-loop system is *marginally* stable.
- The feedback system oscillates at the corresponding frequency.
- Closed-loop poles are the roots of $s^3 + s^2 + 2s + 2 = 0$, i.e.,

$$s = \{-1, \pm j\sqrt{2}\}$$

$k = 1.2$:







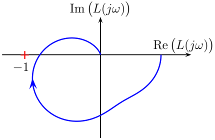
- Closed-loop system is *unstable*.
- The oscillations grow.
- Closed-loop poles are the roots of $s^3 + s^2 + 2s + 2.2 = 0$, i.e.,

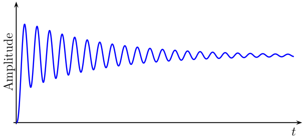
$$s = \{-1.0639, 0.0319 \pm j1.4377\}$$

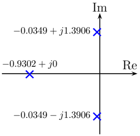
Stability Margins

$L(j\omega)$ coming close to the -1 point without encircling it is undesirable for two reasons:

1. It implies that a closed-loop pole will be close to the imaginary axis and that the closed-loop system will be oscillatory, e.g.,

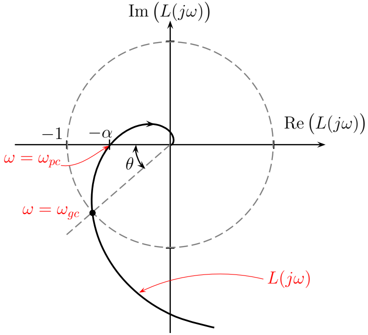






2. If our plant $P(s)$ is the transfer function of an inaccurate model, then the "true" Nyquist diagram might actually encircle -1.

The **gain margin** and the **phase margin** are measures of how close the return ratio $L(j\omega)$ gets to -1.



The **gain margin** (GM) is a measure of how much the gain of $L(s)$ can be increased before the closed-loop system becomes unstable.

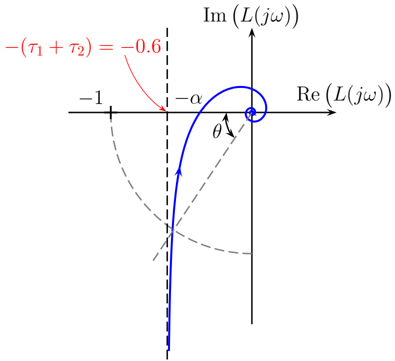
$$GM = \frac{1}{\alpha} = \frac{1}{|L(j\omega_{pc})|}$$

The **phase margin** (PM) is a measure of how much phase lag can be added to $L(s)$ before the closed-loop system becomes unstable.

$$PM = \theta = \pi + \angle L(j\omega_{gc})$$

Example:

Let $L(s) = \dfrac{e^{-s\tau_1}}{s(1+s\tau_2)}$, with $\tau_1 = 0.5$ and $\tau_2 = 0.1$.



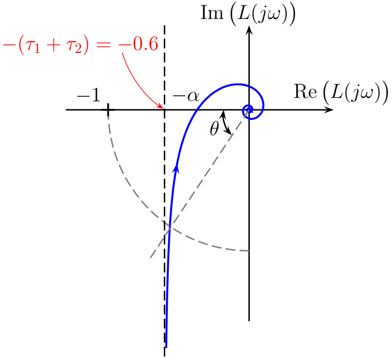
$$GM = \frac{1}{\alpha} = \frac{1}{0.3681} = 2.7164$$

$$PM = \theta = 55.8^\circ$$

- **Nyquist stability theorem** and **stability margins** applied to a system defined by a non-rational function (delay parameter).
- Phase margin can be extracted analytically (as an exercise).
- Gain margin can be approximated (lower bounded) analytically (as an exercise).

Example:

Let $L(s) = \dfrac{e^{-s\tau_1}}{s(1+s\tau_2)}$, with $\tau_1 = 0.5$ and $\tau_2 = 0.1$.



$$GM = \frac{1}{\alpha} = \frac{1}{0.3681} = 2.7164$$

$$PM = \theta = 55.8^\circ$$

- **Nyquist stability theorem** and **stability margins** applied to a system defined by a non-rational function (delay parameter).
- Phase margin can be extracted analytically (as an exercise).
- Gain margin can be approximated (lower bounded) analytically (as an exercise).

How much extra delay can be added to the system before it becomes unstable?

Learning outcomes...

...the student will:

- Learn how to plot the *open-loop* frequency response of the return ratio $L(s)$, with the imaginary part $\mathrm{Im}\left(L(j\omega)\right)$ plotted against the real part $\mathrm{Re}\left(L(j\omega)\right)$ on an Argand diagram.
- Use the Nyquist diagram to ascertain stability of the *closed-loop* system.
- Infer more detailed information about the behavior of the closed-loop system:
  - (a) No need to explicitly compute the poles of the system, so it can be applied to systems defined by non-rational functions.
  - (b) Relatively easy to see how changing $C(s)$ affects $L(s)$, but difficult to see how $C(s)$ affects $L(s)/(1+L(s))$ directly.
  - (c) *Gain* and *phase* margins measure how close the closed-loop system is to instability.