MEC-E8003

Beam, Plate and Shell Models

2023

WEEK 10: KINEMATICS

2 KINEMATICS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on kinematics:

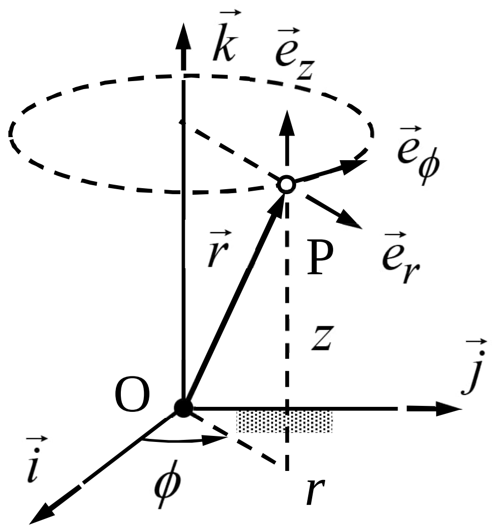
- Material coordinate system. Vectors, basis vector derivatives, and gradient operator in the polar, cylindrical, and spherical material coordinate systems.

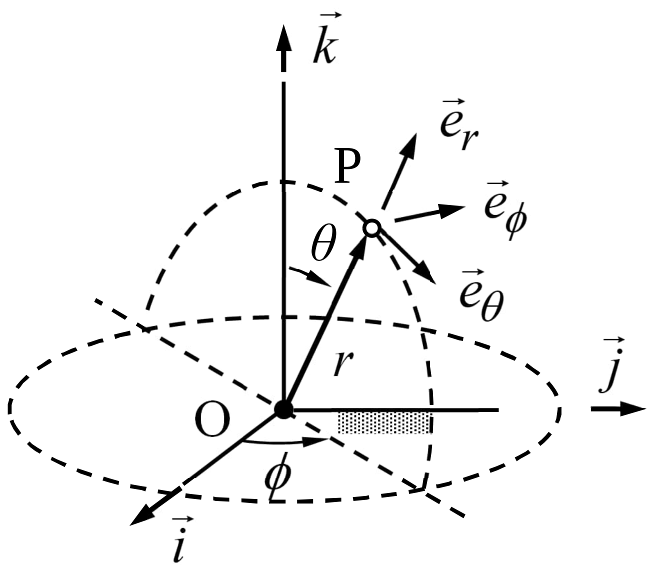
- Basis vectors, basis vector derivatives, and gradient operator in the beam and shell material coordinate systems.

- Curvature of curves and surfaces.

2.1 COORDINATE SYSTEMS

In solid mechanics, particles of a body (a closed system of particles) are identified by coordinates of the initial geometry. Equilibrium equations etc. can be written for any selection of the material coordinates, but a clever selection may simplify the setting.





A Cartesian (x, y, z) coordinate system with known derivatives of the basis vector, gradient operator etc. is always needed as a reference system.

CURVILINEAR MATERIAL COORDINATE SYSTEM

Position vector: $\vec{r}(\alpha,\beta,\gamma)=\begin{Bmatrix} x(\alpha,\beta,\gamma) \\ y(\alpha,\beta,\gamma) \\ z(\alpha,\beta,\gamma) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$, $\quad \begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \\ \vec{h}_\gamma \end{Bmatrix}=\begin{Bmatrix} \partial\vec{r}/\partial\alpha \\ \partial\vec{r}/\partial\beta \\ \partial\vec{r}/\partial\gamma \end{Bmatrix}=\left[H\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$

Basis vectors: $\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}=\begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{h}_\gamma / h_\gamma \end{Bmatrix}=\left[F\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$, $\quad \dfrac{\partial}{\partial\eta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}=(\dfrac{\partial}{\partial\eta}\left[F\right])\left[F\right]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}$

Gradient: $\nabla=\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} \left[F\right]^{-\mathrm{T}}\left[H\right]^{-1}\begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix}$,

BASIS VECTORS

The basis vectors of a Cartesian coordinate system are constants. Starting with the position vector $\vec{r}(\alpha,\beta,\gamma) = x(\alpha,\beta,\gamma)\vec{i} + y(\alpha,\beta,\gamma)\vec{j} + z(\alpha,\beta,\gamma)\vec{k}$ of particle (α,β,γ) and using definitions $\vec{h}_\alpha = \partial\vec{r}/\partial\alpha$, $\vec{h}_\beta = \partial\vec{r}/\partial\beta$, $\vec{h}_\gamma = \partial\vec{r}/\partial\gamma$ (h_α, h_β, h_γ are the lengths or, later, the scaling coefficients)

Basis vectors: $$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{h}_\gamma / h_\gamma \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}.$$

Basis vector derivatives: $$\frac{\partial}{\partial\eta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = (\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} \quad \eta \in \{\alpha,\beta,\gamma\}$$

The starting point is the position vector of a material point in the reference system $\vec{r}(\alpha,\beta,\gamma) = x(\alpha,\beta,\gamma)\vec{i} + y(\alpha,\beta,\gamma)\vec{j} + z(\alpha,\beta,\gamma)\vec{k}$ expressed in terms of (α,β,γ) identifying the particles. Basis vectors of the curvilinear $(\alpha,\beta,\gamma)-$ system

$$\begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \\ \vec{h}_\gamma \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r}/\partial\alpha \\ \partial\vec{r}/\partial\beta \\ \partial\vec{r}/\partial\gamma \end{Bmatrix} = \left[H\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ and } \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha/h_\alpha \\ \vec{h}_\beta/h_\beta \\ \vec{h}_\gamma/h_\gamma \end{Bmatrix} = \left[F\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \left[F\right]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}$$

where $h_\alpha = \left|\vec{h}_\alpha\right| \equiv \sqrt{\vec{h}_\alpha \cdot \vec{h}_\alpha}$, $h_\beta = \left|\vec{h}_\beta\right|$, and $h_\gamma = \left|\vec{h}_\gamma\right|$ are the scaling coefficients.

As basis vectors of the Cartesian reference coordinate system are constants, the derivatives of the basis vectors of the curvilinear (α,β,γ) coordinate system with respect to $\eta \in \{\alpha,\beta,\gamma\}$ become

$$\frac{\partial}{\partial \eta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \frac{\partial}{\partial \eta}\left[F\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \left(\frac{\partial}{\partial \eta}\left[F\right]\right)\left[F\right]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}.$$

In the last form, the relationship between the basis vectors is used the other way around to have the derivatives in the basis of the curvilinear system.

GRADIENT OPERATOR

As a vector, gradient is invariant with respect to the coordinate system. Selection of the material coordinates and the related basis vectors affect, however, the representation

$$(x,y,z):\quad \nabla=\begin{Bmatrix}\vec{e}_x\\ \vec{e}_y\\ \vec{e}_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix}=\vec{e}_x\frac{\partial}{\partial x}+\vec{e}_y\frac{\partial}{\partial y}+\vec{e}_z\frac{\partial}{\partial z}\ ,$$

$$(\alpha,\beta,\gamma):\ \nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix}\partial/\partial\alpha\\ \partial/\partial\beta\\ \partial/\partial\gamma\end{Bmatrix}\text{ where }[H]=\begin{bmatrix}\partial x/\partial\alpha & \partial y/\partial\alpha & \partial z/\partial\alpha\\ \partial x/\partial\beta & \partial y/\partial\beta & \partial z/\partial\beta\\ \partial x/\partial\gamma & \partial y/\partial\gamma & \partial z/\partial\gamma\end{bmatrix}.$$

Notice that $[F]$ and $[H]$ differ only in the scaling of the rows!

Using the chain rule, the relationships between coordinates and basis vectors and the (coordinate system) invariance of the gradient operator (it is a vector)

$$\begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} = \begin{bmatrix} \partial x/\partial\alpha & \partial y/\partial\alpha & \partial z/\partial\alpha \\ \partial x/\partial\beta & \partial y/\partial\beta & \partial z/\partial\beta \\ \partial x/\partial\gamma & \partial y/\partial\gamma & \partial z/\partial\gamma \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} = [H] \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} \;\Rightarrow$$

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} \;\Leftrightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix}. \;\Leftarrow$$

Matrices $[H]$ and $[F]$ differ only in scaling of the rows. Denoting the diagonal matrix of the scaling coefficient by $[h] = diag\{h_\alpha, h_\beta, h_\gamma\}$ it holds $[h][F] = [H]$. Further, in an orthonormal coordinate system $[F]^{-1} = [F]^{\mathrm{T}}$ so

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix} = \vec{e}_\alpha \frac{1}{h_\alpha} \frac{\partial}{\partial\alpha} + \vec{e}_\beta \frac{1}{h_\beta} \frac{\partial}{\partial\beta} + \vec{e}_\gamma \frac{1}{h_\gamma} \frac{\partial}{\partial\gamma} .$$

Therefore, it is enough to know the scaling coefficients.

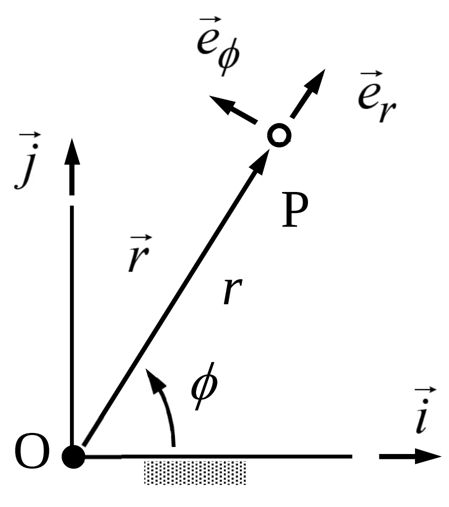
POLAR COORDINATES (r,ϕ)

In a curvilinear rectangular Polar coordinate system, a particle is identified by its distance r from the origin and angle ϕ from a chosen line. Basis vectors, their derivatives, and the gradient operator are given by mapping $\vec{r}(r,\phi) = r\cos\phi\vec{i} + r\sin\phi\vec{j}$:

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix},$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix} \quad \text{(otherwise zeros),}$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}.$$



The derivatives follow from the generic expression or in a more clear manner from steps (just to emphasize the idea)

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Rightarrow$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \left(\frac{\partial}{\partial\phi} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\right) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \Rightarrow$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}. \quad \Leftarrow$$

In writing the gradient expression, one needs the relationships between basis and partial derivatives in a Cartesian and polar coordinate systems:

$$\begin{Bmatrix} \partial / \partial r \\ \partial / \partial \phi \end{Bmatrix} = \begin{bmatrix} \partial x / \partial r & \partial y / \partial r \\ \partial x / \partial \phi & \partial y / \partial \phi \end{bmatrix} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix} = [H] \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}.$$

Using the vector (operator) invariance with respect to the coordinate system

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial r \\ \partial / \partial \phi \end{Bmatrix} \Rightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \right)^{-1} \begin{Bmatrix} \partial / \partial r \\ \partial / \partial \phi \end{Bmatrix} \Leftrightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} \begin{Bmatrix} \partial / \partial r \\ \partial / \partial \phi \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}. \quad \Leftarrow$$

EXAMPLE Derive the component forms of the balance law $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in the polar coordinate system when stress and distributed force

$$\vec{\vec{\sigma}}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{rr} & \sigma_{r\phi}\\ \sigma_{\phi r} & \sigma_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} \text{ and } \vec{f}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_r\\ f_\phi\end{Bmatrix},$$

respectively. Derivatives of the basis vectors and the gradient operator of the polar coordinate system are

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix} \text{ and } \nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}.$$

Answer $\frac{1}{r}[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}]+f_r=0$ and $\frac{1}{r}[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}]+f_\phi=0$

In polar coordinate system basis vectors depend on the angular coordinate. First, let us expand the stress divergence and consider the terms one-by-one by keeping the order of the basis vectors and position of the inner product:

$$\nabla \cdot \vec{\vec{\sigma}} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\sigma_{rr}\vec{e}_r\vec{e}_r + \sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \quad \Leftrightarrow$$

$$\nabla \cdot \vec{\vec{\sigma}} = \vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{rr}\vec{e}_r\vec{e}_r + \vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi +$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{rr}\vec{e}_r\vec{e}_r + \vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi \, .$$

Next, by considering the terms one-by-one by keeping the order of the basis vectors, position of the inner product, and taking into account the non-zero derivatives $\partial\vec{e}_r / \partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi / \partial\phi = -\vec{e}_r$,

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{rr}\vec{e}_r\vec{e}_r = \frac{\partial \sigma_{rr}}{\partial r}(\vec{e}_r \cdot \vec{e}_r)\vec{e}_r = \frac{\partial \sigma_{rr}}{\partial r}\vec{e}_r, \qquad \text{(basis vectors are orthonormal)}$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi = \frac{\partial \sigma_{r\phi}}{\partial r}(\vec{e}_r \cdot \vec{e}_r)\vec{e}_\phi = \frac{\partial \sigma_{r\phi}}{\partial r}\vec{e}_\phi,$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r = \frac{\partial \sigma_{\phi r}}{\partial r}(\vec{e}_r \cdot \vec{e}_\phi)\vec{e}_r = 0,$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi = \frac{\partial \sigma_{\phi\phi}}{\partial r}(\vec{e}_r \cdot \vec{e}_\phi)\vec{e}_\phi = 0,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{rr}\vec{e}_r\vec{e}_r = \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_r)\vec{e}_r + \frac{1}{r}\sigma_{rr}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_r}{\partial \phi})\vec{e}_r + \frac{1}{r}\sigma_{rr}(\vec{e}_\phi \cdot \vec{e}_r)\frac{\partial \vec{e}_r}{\partial \phi} = \frac{1}{r}\sigma_{rr}\vec{e}_r,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi = \frac{1}{r}\frac{\partial \sigma_{r\phi}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_r)\vec{e}_\phi + \frac{1}{r}\sigma_{r\phi}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_r}{\partial \phi})\vec{e}_\phi + \frac{1}{r}\sigma_{r\phi}(\vec{e}_\phi \cdot \vec{e}_r)\frac{\partial \vec{e}_\phi}{\partial \phi} = \frac{1}{r}\sigma_{r\phi}\vec{e}_\phi,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r = \frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial\phi}(\vec{e}_\phi\cdot\vec{e}_\phi)\vec{e}_r + \frac{1}{r}\sigma_{\phi r}(\vec{e}_\phi\cdot\frac{\partial\vec{e}_\phi}{\partial\phi})\vec{e}_r + \frac{1}{r}\sigma_{\phi r}(\vec{e}_\phi\cdot\vec{e}_\phi)\frac{\partial\vec{e}_r}{\partial\phi}$$

$$= \frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial\phi}\vec{e}_r + \frac{1}{r}\sigma_{\phi r}\vec{e}_\phi,$$

$$\vec{e}_\phi \cdot \frac{\partial}{r\partial\phi}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi = \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}(\vec{e}_\phi\cdot\vec{e}_\phi)\vec{e}_\phi + \frac{1}{r}\sigma_{\phi\phi}(\vec{e}_\phi\cdot\frac{\partial\vec{e}_\phi}{\partial\phi})\vec{e}_\phi + \frac{1}{r}\sigma_{\phi\phi}(\vec{e}_\phi\cdot\vec{e}_\phi)\frac{\partial\vec{e}_\phi}{\partial\phi}$$

$$= \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\vec{e}_\phi - \frac{1}{r}\sigma_{\phi\phi}\vec{e}_r,$$

Finally, by combining the terms

$$\nabla\cdot\vec{\sigma} = \frac{\partial\sigma_{rr}}{\partial r}\vec{e}_r + \frac{\partial\sigma_{r\phi}}{\partial r}\vec{e}_\phi + \frac{1}{r}\sigma_{rr}\vec{e}_r + \frac{1}{r}\sigma_{r\phi}\vec{e}_\phi + \frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial\phi}\vec{e}_r + \frac{1}{r}\sigma_{\phi r}\vec{e}_\phi + \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\vec{e}_\phi - \frac{1}{r}\sigma_{\phi\phi}\vec{e}_r$$

$$\nabla \cdot \vec{\vec{\sigma}} = \frac{1}{r}\left(r\frac{\partial \sigma_{rr}}{\partial r} + \sigma_{rr} + \frac{\partial \sigma_{\phi r}}{\partial \phi} - \sigma_{\phi\phi}\right)\vec{e}_r + \frac{1}{r}\left(r\frac{\partial \sigma_{r\phi}}{\partial r} + \sigma_{r\phi} + \sigma_{\phi r} + \frac{\partial \sigma_{\phi\phi}}{\partial \phi}\right)\vec{e}_\phi \quad \Leftrightarrow$$

$$\nabla \cdot \vec{\vec{\sigma}} = \frac{1}{r}\left[\frac{\partial (r\sigma_{rr})}{\partial r} + \frac{\partial \sigma_{\phi r}}{\partial \phi} - \sigma_{\phi\phi}\right]\vec{e}_r + \frac{1}{r}\left[\frac{\partial (r\sigma_{r\phi})}{\partial r} + \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \sigma_{\phi r}\right]\vec{e}_\phi .$$

With the distributed force $\vec{f} = f_r\vec{e}_r + f_\phi\vec{e}_\phi$, the local form of the momentum balance $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0$ in the polar coordinate system

$$\frac{1}{r}\left[\frac{\partial (r\sigma_{rr})}{\partial r} + \frac{\partial \sigma_{\phi r}}{\partial \phi} - \sigma_{\phi\phi} + rf_r\right]\vec{e}_r + \frac{1}{r}\left[\frac{\partial (r\sigma_{r\phi})}{\partial r} + \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \sigma_{\phi r} + rf_\phi\right]\vec{e}_\phi = 0.$$

EXAMPLE The small strain measure is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor (a) in Cartesian coordinate system and (b) in the polar system.

Answer (a) $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}$, and $\varepsilon_{xy} = \varepsilon_{yx} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y})$

(b) $\varepsilon_{rr} = \dfrac{\partial u_r}{\partial r}$, $\varepsilon_{\phi\phi} = \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial \phi} + u_r)$, and $\varepsilon_{r\phi} = \varepsilon_{\phi r} = \dfrac{1}{2}(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \phi} - \dfrac{u_\phi}{r} + \dfrac{\partial u_\phi}{\partial r})$

In Cartesian system, $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$, therefore

$$\nabla\vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})(u_x\vec{i} + u_y\vec{j}) = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{i}\vec{j}\frac{\partial u_y}{\partial x} + \vec{j}\vec{i}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}) \Rightarrow$$

$$(\nabla\vec{u})_{\mathrm{c}} = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{i}\frac{\partial u_y}{\partial x} + \vec{i}\vec{j}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}$$

giving

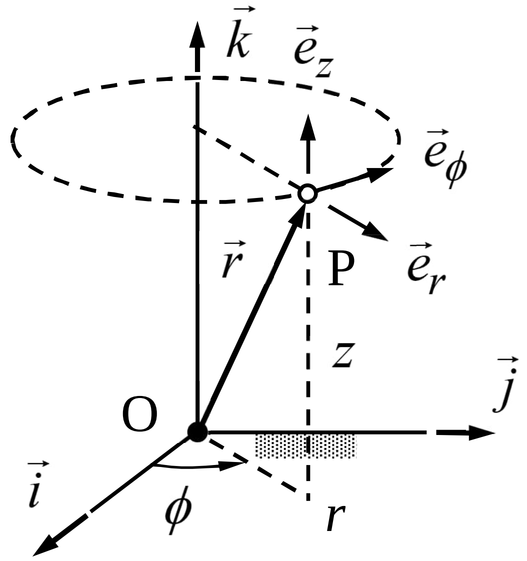
$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y} + \vec{i}\vec{j}\frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) + \vec{j}\vec{i}\frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}).$$

In polar coordinates $\vec{u} = u_r\vec{e}_r + u_\phi\vec{e}_\phi$, and $\nabla = \vec{e}_r\partial/\partial r + \vec{e}_\phi\partial/(r\partial\phi)$, $\partial\vec{e}_r/\partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi/\partial\phi = -\vec{e}_r$. Otherwise, calculation follows the steps used with the Cartesian coordinate system (one of the exercise problems).

CYLINDRICAL COORDINATES (r, ϕ, z)

A particle is identified by its distance r from the z-axis origin, angle ϕ from the x-axis and distance z from the xy-plane. Mapping $\vec{r} = r\cos\phi\vec{i} + r\sin\phi\vec{j} + z\vec{k}$ gives

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



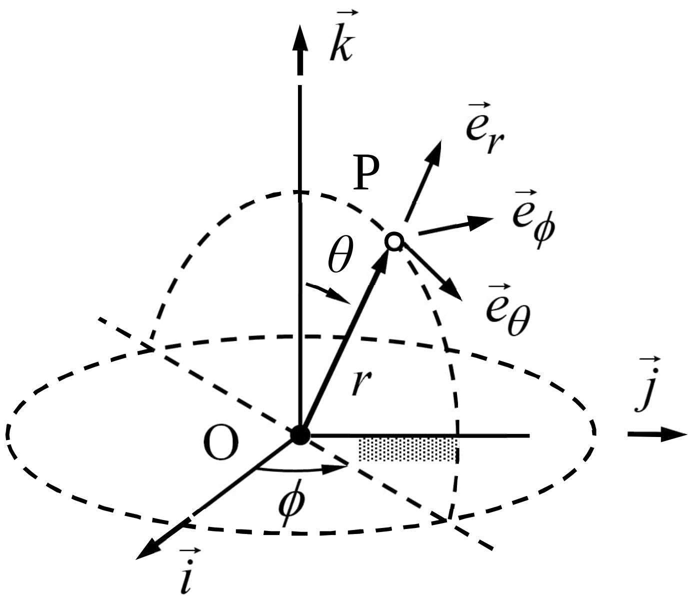
$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} \text{ otherwise zeros,}$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}.$$

SPHERICAL COORDINATES (θ, ϕ, r)

A particle is identified by its distance r, angle ϕ from the *x*-axis, and angle θ from the *z*-axis. Mapping $\vec{r}(\theta,\phi,r) = r(\sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k})$, gives

$$\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} \cos\theta\vec{e}_\phi \\ -\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta \\ \sin\theta\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} -\vec{e}_r \\ 0 \\ \vec{e}_\theta \end{Bmatrix},$$

$$\nabla = \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} + \vec{e}_r \frac{\partial}{\partial r}.$$

According to the generic recipe (here $\mathrm{c} \sim \cos$ and $\mathrm{s} \sim \sin$)

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = (\frac{\partial}{\partial r}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix},$$

$$\frac{\partial}{\partial \phi}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = (\frac{\partial}{\partial \phi}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{bmatrix}0 & \mathrm{c}\theta & 0\\ -\mathrm{c}\theta & 0 & -\mathrm{s}\theta\\ 0 & \mathrm{s}\theta & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}\mathrm{c}\theta\,\vec{e}_\phi\\ -\mathrm{s}\theta\vec{e}_r - \mathrm{c}\theta\vec{e}_\theta\\ \mathrm{s}\theta\vec{e}_\phi\end{Bmatrix},$$

$$\frac{\partial}{\partial \theta}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = (\frac{\partial}{\partial \theta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}-\vec{e}_r\\ 0\\ \vec{e}_\theta\end{Bmatrix}. \Leftarrow$$

2.2 CURVES AND SURFACES

The solution domain Ω of an engineering (beam or plate) model has usually lower dimension than the body $V \in \mathbb{R}^3$. The representation of the domain embedded in $\mathbb{R}^3$ may be mid-line or mid-curve of curve of beam or mid-plane or mid-surface of plate)

Curve: $\vec{r}_0(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j} + z(\alpha)\vec{k} \qquad \alpha \in \Omega \subset \mathbb{R}$ 1 parameter

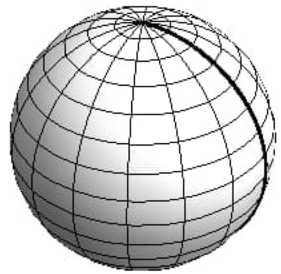
Surface: $\vec{r}_0(\alpha,\beta) = x(\alpha,\beta)\vec{i} + y(\alpha,\beta)\vec{j} + z(\alpha,\beta)\vec{k} \quad (\alpha,\beta) \in \Omega \subset \mathbb{R}^2$ 2 parameters

Shape of a mid-curve is defined by a one-parameter mapping and a mid-surface by a two-parameter mapping. In MEC-E8003, the coordinate curves of surfaces (defined by constant values of α or β**)** are assumed to be orthogonal (just to simplify the setting).

SOME MAPPINGS

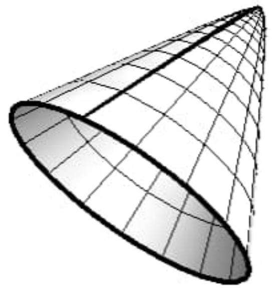
Coil $\vec{r}_0(\phi) = \vec{i}R\cos\phi + \vec{j}R\sin\phi + \vec{k}h\dfrac{\phi}{2\pi},$

Cylinder $\vec{r}_0(z,\phi) = R(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k}z$



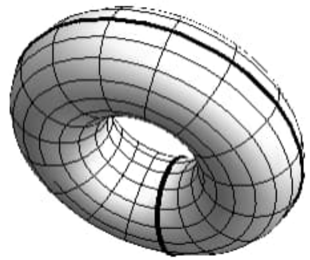
Cone $\vec{r}_0(z,s) = R(z)(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k}z$

Sphere $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta)$



Ellipsoid $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \varepsilon\vec{k}\cos\theta)$

Hyperboloid $\vec{r}_0(\phi,\theta) = R(\vec{i}\sinh\theta\cos\phi + \vec{j}\sin h\theta\sin\phi - \varepsilon\vec{k}\cosh\theta)$

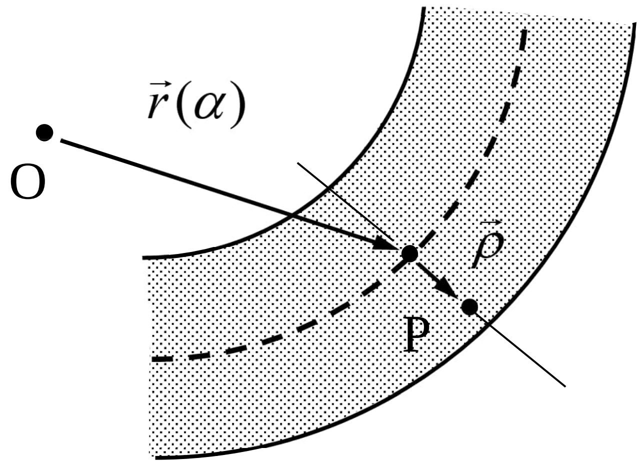


Torus $\vec{r}_0(\phi,\theta) = \vec{i}\cos\phi(R + r\cos\theta) + \vec{j}\sin\phi(R + r\cos\theta) + \vec{k}r\sin\theta$

Mid-curve or mid-surface mapping identifies the particles on the mid-curve or mid-surface. Identification of all particles (P in the figure) of a thin body requires also the relative position vector $\vec{\rho}$:

Beam mapping: $\vec{r}(\alpha,n,b)=\vec{r}_0(\alpha)+\overbrace{n\vec{e}_n(\alpha)+b\vec{e}_b(\alpha)}^{\text{relative}}$

Shell mapping: $\vec{r}(\alpha,\beta,n)=\vec{r}_0(\alpha,\beta)+\underbrace{n\vec{e}_n(\alpha,\beta)}_{\text{relative}}$



The mapping for the mid-curve or surface is used to define the basis vectors. In MEC-E8003 basis vectors are orthonormal to keep the setting as simple as possible (curved geometry induces some complications anyway)!

BEAM COORDINATES (s,n,b)

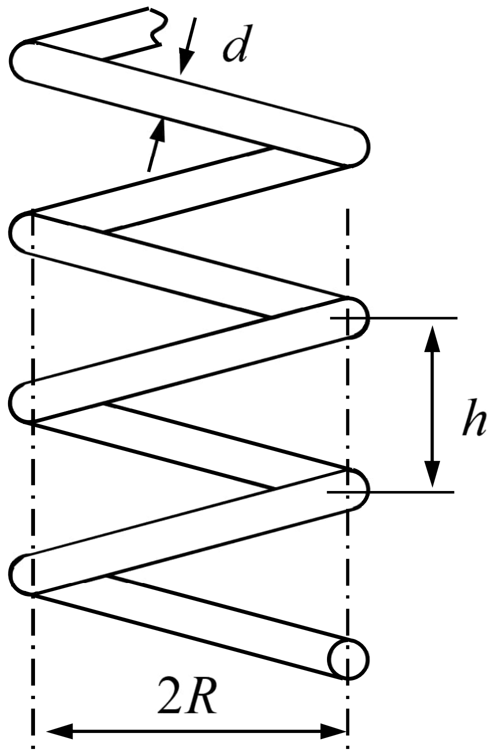
Particle is identified by distance s along the mid-curve and distances (n,b) from the curve.

Mapping $\vec{r}(s,n,b) = \vec{r}_0(s) + n\vec{e}_n(s) + b\vec{e}_b(s)$ gives

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r}_0 / \partial s \\ (\partial\vec{e}_s / \partial s) / \left|\partial\vec{e}_s / \partial s\right| \\ \vec{e}_s \times \vec{e}_n \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \begin{Bmatrix} \partial\vec{r} / \partial s \\ \partial\vec{r} / \partial n \\ \partial\vec{r} / \partial b \end{Bmatrix} = [H]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

$$\frac{\partial}{\partial s}\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \kappa\vec{e}_n \\ -\kappa\vec{e}_s + \tau\vec{e}_b \\ -\tau\vec{e}_n \end{Bmatrix},$$

$$\nabla = \frac{1}{1-n\kappa}\vec{e}_s[\frac{\partial}{\partial s} + \tau(b\frac{\partial}{\partial n} - n\frac{\partial}{\partial b})] + \vec{e}_n\frac{\partial}{\partial n} + \vec{e}_b\frac{\partial}{\partial b}.$$



Beam (s,n,b) coordinate system is curvilinear and orthonormal. Therefore the matrix of the basis vector derivatives is anti-symmetric (why?) and expressible in form (Serret-Frenet formulas in literature)

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\\vec{e}_n\\\vec{e}_b\end{Bmatrix}=(\frac{\partial}{\partial s}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_s\\\vec{e}_n\\\vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa_b & -\kappa_n\\-\kappa_b & 0 & \kappa_s\\\kappa_n & -\kappa_s & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\\vec{e}_n\\\vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\kappa\vec{e}_n\\-\kappa\vec{e}_s+\tau\vec{e}_b\\-\tau\vec{e}_n\end{Bmatrix}$$

containing geometrical quantities $\kappa_s=\tau$, $\kappa_n=0$, and $\kappa_b=\kappa=1/R$. Explicit expressions for *curvature* κ and *torsion* τ require an explicit form of $\vec{r}_0(s)$ or $[F]$.

The gradient operator at a generic point *in terms of the basis vectors at the mid-curve* is based on $\vec{r}=\vec{r}_0+n\vec{e}_n+b\vec{e}_b$ the beam is given by ($[H]$ follows from $\vec{r}$ and $[F]$ follows from $\vec{r}_0$

$$\begin{Bmatrix} \partial\vec{r}/\partial s \\ \partial\vec{r}/\partial n \\ \partial\vec{r}/\partial b \end{Bmatrix} = \begin{bmatrix} 1-\kappa n & -\tau b & \tau n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 1-\kappa n & -\tau b & \tau n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [H] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

As the curvilinear coordinate system is orthonormal so $[F]^{\mathrm{T}} = [F]^{-1}$, the generic formula for gradient gives

$$\nabla = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \left([H][F]^{\mathrm{T}}\right)^{-1} \begin{Bmatrix} \partial/\partial s \\ \partial/\partial n \\ \partial/\partial b \end{Bmatrix} = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1-\kappa n & -\tau b & \tau n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \partial/\partial s \\ \partial/\partial n \\ \partial/\partial b \end{Bmatrix}.$$

SHELL COORDINATES (α,β,n)

A particle is identified by mid-surface position (α,β) (generalized coordinates) and distance n in the normal direction. Mapping $\vec{r}(\alpha,\beta,n)=\vec{r}_0(\alpha,\beta)+n\vec{e}_n(\alpha,\beta)$ gives

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r}_0/\partial\alpha)/|\partial\vec{r}_0/\partial\alpha| \\ (\partial\vec{r}_0/\partial\beta)/|\partial\vec{r}_0/\partial\beta| \\ \vec{e}_\alpha\times\vec{e}_\beta \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \begin{Bmatrix} \partial\vec{r}/\partial\alpha \\ \partial\vec{r}/\partial\beta \\ \partial\vec{r}/\partial n \end{Bmatrix} = [H]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

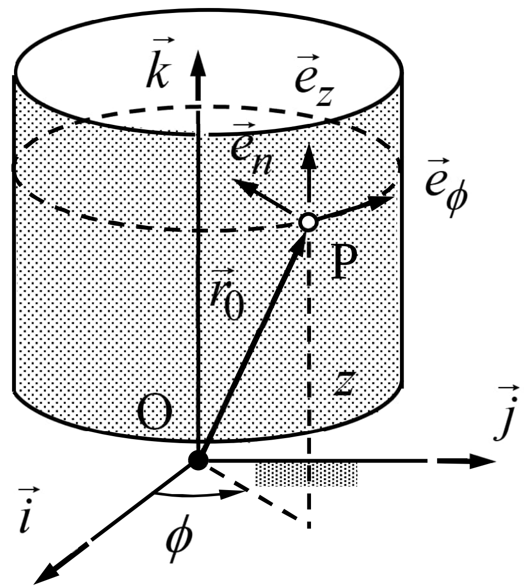
$$\frac{\partial}{\partial\eta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = (\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} \; \eta\in\{\alpha,\beta,n\} \text{ and } \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial n \end{Bmatrix}$$

In MEC-E8003, the mapping $\vec{r}_0(\alpha,\beta)$ is restricted by orthogonality condition $\vec{e}_\alpha\cdot\vec{e}_\beta=0$.

CYLINDRICAL SHELL (z,ϕ,n)

A particle is identified by mid-surface coordinates (z,ϕ) and distance n in the normal direction (inwards). Mid-surface mapping $\vec{r}_0(\phi,z)=\vec{i}R\cos\phi+\vec{j}R\sin\phi+\vec{k}z$ gives

$$\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0 & 0 & 1\\ -\sin\phi & \cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},$$



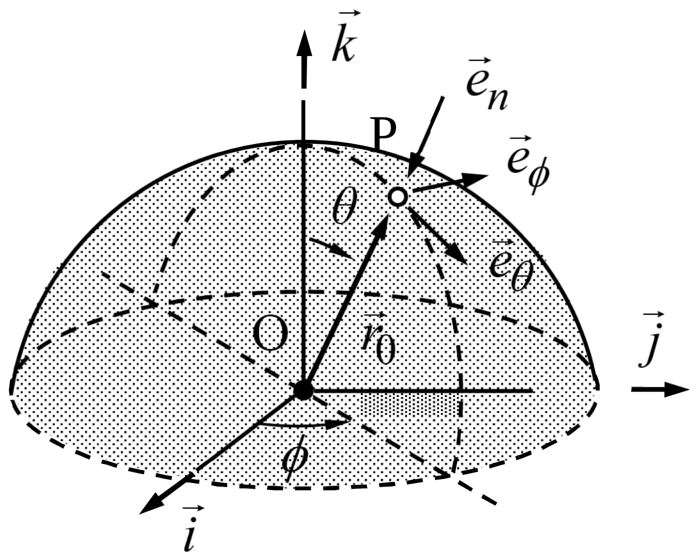
$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\phi\end{Bmatrix}\text{ zeros otherwise,}$$

$$\nabla=\vec{e}_z\frac{\partial}{\partial z}+\frac{R}{R-n}\frac{1}{R}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}.$$

SPHERICAL SHELL (ϕ, θ, n)

A particle is identified by mid-surface position (ϕ, θ) and distance n in the normal direction (inwards). Mid-surface mapping $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta)$:

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta \\ \cos\theta\vec{e}_\phi \\ -\sin\theta\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix},$$

$$\nabla = \frac{R}{R-n}\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{R}{R-n}\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta} + \vec{e}_n\frac{\partial}{\partial n}.$$

2.3 CURVATURE

Curvature is the amount by which surface or curve embedded in $\mathbb{R}^3$ deviates from being *flat* or *straight*. The radius of curvature $R = 1/\kappa$ of a curve is given by the best fitting circle. Curvature of a surface at a point depends on the direction of a curve through that point.

Curvature: $\overset{\leftrightarrow}{\kappa}_{\mathrm{c}} = \nabla_0 \vec{e}_n$ Gradient at the mid-curve or mid-surface !

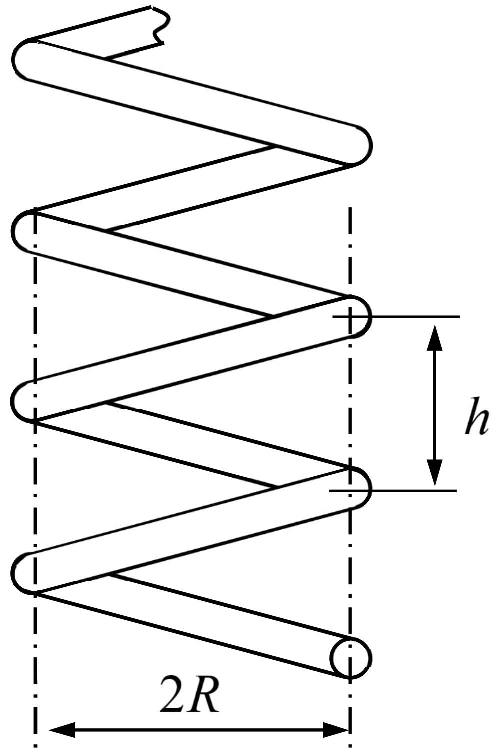
Principal curvatures: $(\kappa_1, \vec{n}_1)$ and $(\kappa_2, \vec{n}_2)$ such that $\overset{\leftrightarrow}{\kappa} \cdot \vec{n} = \kappa \vec{n}$

Gaussian curvature: $K = \det[\kappa] = \kappa_1 \kappa_2$ Curvature measure!

Mean curvature: $H = \dfrac{1}{2} \nabla \cdot \vec{e}_n = \dfrac{1}{2} \overset{\leftrightarrow}{I} : \overset{\leftrightarrow}{\kappa} = \dfrac{1}{2}(\kappa_1 + \kappa_2)$ Another curvature measure!

CURVATURE AND TORSION OF BEAM

The radius of curvature $R = 1/\kappa$ at a point is given by the best fitting circle. Torsion τ describes the rate of rotation of $\vec{e}_n$ and $\vec{e}_b$ around the mid-curve (change of rotation angle divided by change of the mid-curve coordinate s)



Circular: $\kappa = \dfrac{1}{R}$ and $\tau = 0$

Twisted beam: $\kappa = 0$ and $\tau = \dfrac{2\pi}{h}$

Coil: $\kappa = \dfrac{R}{h^2 + R^2}$ and $\tau = \dfrac{h}{h^2 + R^2}$

A circular beam of radius R has zero torsion. The basis vectors of (x,y,z) and (s,n,b) coordinate systems differ by rotation with respect to the normal direction to the plane of circle (z here). With distance s along the mid-curve and $\phi = s/R$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow \quad \kappa = \vec{e}_n \cdot \frac{\partial}{\partial s}\vec{e}_s = \frac{1}{R}. \quad \Leftarrow$$

A twisted beam has zero curvature. The basis vectors of (x,y,z) and (s,n,b) differ by rotation along the $x-$axis. With notation $\omega = 2\phi/h$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega s & \sin\omega s \\ 0 & -\sin\omega s & \cos\omega s \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow \quad \tau = \vec{e}_b \cdot \frac{\partial}{\partial s}\vec{e}_n = \frac{2\pi}{h}. \quad \Leftarrow$$

EXAMPLE A planar curve in $xy-$ plane is defined by mapping

(a) $\vec{r}_0(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j}$ (generic parametric form of a planar curve)

(b) $\vec{r}_0(x) = x\vec{i} + y(x)\vec{j}$

Derive the curvature tensor $\overleftrightarrow{\kappa}_\mathrm{c} = \nabla_0 \vec{e}_n$.

Answer: (a) $\overleftrightarrow{\kappa} = -\vec{e}_\alpha \vec{e}_\alpha \dfrac{|y'x'' - x'y''|}{(x'^2 + y'^2)^{3/2}}$ (b) $\overleftrightarrow{\kappa} = -\vec{e}_x \vec{e}_x \dfrac{|y''|}{(1 + y'^2)^{3/2}}$

To use the definition, one needs the derivatives of the basis vectors and also the gradient operator of the curvilinear (α, n, b) system at $n = b = 0$. With the Lagrange's notation of derivative with respect to α

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_n / h_n \\ \vec{e}_\alpha \times \vec{e}_n \end{Bmatrix} = \begin{bmatrix} x' / h_\alpha & y' / h_\alpha & 0 \\ \mp y' / h_\alpha & \pm x' / h_\alpha & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$

where $h_\alpha = \sqrt{x'^2 + y'^2}$ and $h_n = \dfrac{|y'x'' - x'y''|}{x'^2 + y'^2}$.

The derivatives of the basis vectors follow from the generic expression

$$\frac{\partial}{\partial\alpha}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=(\frac{\partial}{\partial\alpha}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=h_n\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=h_n\begin{Bmatrix}\vec{e}_n\\ -\vec{e}_\alpha\\ 0\end{Bmatrix}.$$

At the mid-curve, where $n=b=0$, the gradient operator for a curvilinear orthonormal coordinate system of a beam simplifies to $(\kappa=h_n/h_\alpha)$

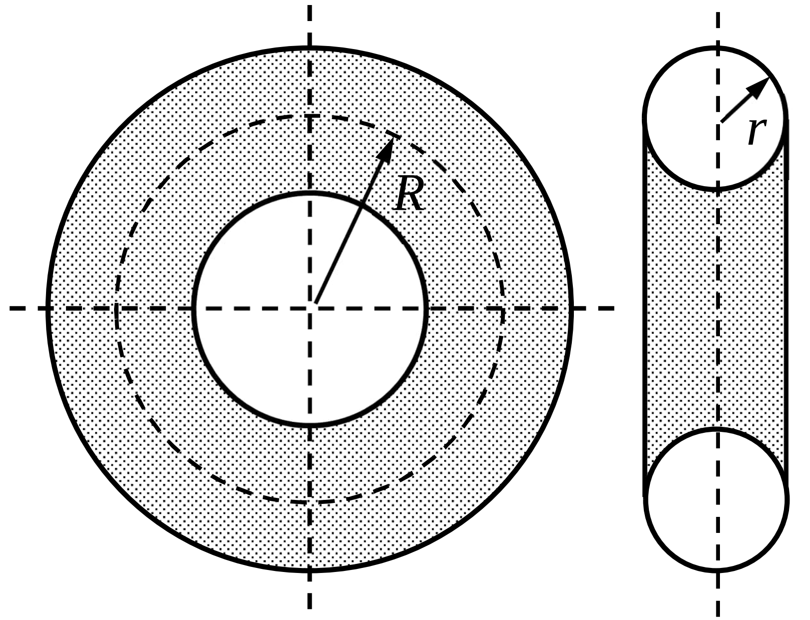
$$\nabla=\vec{e}_\alpha\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}+\vec{e}_n\frac{\partial}{\partial n}+\vec{e}_b\frac{\partial}{\partial b}\qquad\Rightarrow$$

$$\vec{\vec{\kappa}}_{\mathrm{c}}=\nabla\vec{e}_n=\vec{e}_\alpha\frac{1}{h_\alpha}\frac{\partial\vec{e}_n}{\partial\alpha}=-\vec{e}_\alpha\vec{e}_\alpha\frac{h_n}{h_\alpha}=-\vec{e}_\alpha\vec{e}_\alpha\frac{|y'x''-x'y''|}{(x'^2+y'^2)^{3/2}}\qquad\Leftrightarrow$$

$$\vec{\vec{\kappa}}=-\vec{e}_\alpha\vec{e}_\alpha\frac{|y'x''-x'y''|}{(x'^2+y'^2)^{3/2}}\,.$$

EXAMPLE Consider torus surface (donut) having distance R from the center of the tube to the center of the torus and radius r of the tube. Derive the basis vectors, basis vector derivatives, gradient expression, and curvature in (ϕ,θ,n) coordinate system. The mapping defining the geometry, $\phi\in[0,2\pi]$ and $\theta\in[0,2\pi]$, is

$$\vec{r}_0(\phi,\theta)=(R+r\cos\theta)(\vec{i}\cos\phi+\vec{j}\sin\phi)+\vec{k}r\sin\theta .$$



Answer:

$$\nabla=\frac{1}{R+(n+r)\cos\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_\theta\frac{1}{n+r}\frac{\partial}{\partial\theta}+\vec{e}_n\frac{\partial}{\partial n}\ ,\quad \vec{\kappa}=\vec{e}_\phi\vec{e}_\phi\frac{\cos\theta}{R+(n+r)\cos\theta}+\vec{e}_\theta\vec{e}_\theta\frac{1}{n+r}$$

Let us start with the relationship between the basis vectors. Definitions give

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi\sin\theta & -\sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

Since the basis is orthonormal i.e. $[F]^{-1} = [F]^{\mathrm{T}}$, the derivatives of the basis vectors are given by

antisymmetric!

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = (\frac{\partial}{\partial\phi}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & \sin\theta & -\cos\theta \\ -\sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix}$$

$$\frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\left(\frac{\partial}{\partial\theta}[F]\right)[F]^{-1}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix},\quad \text{and } \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=0.$$

The gradient expression in concerned with a generic material point so that the mapping between the curvilinear (ϕ,θ,n) coordinate system and the reference (x,y,z) coordinate system is written as $\vec{r}=\vec{r}_0+n\vec{e}_n$ (the mapping needs to define positions of all the particles of body not just those on the mid-surface). With $\vec{h}_\phi=\partial\vec{r}/\partial\phi$ etc.

$$\begin{Bmatrix}\vec{h}_\phi\\ \vec{h}_\theta\\ \vec{h}_n\end{Bmatrix}=\begin{bmatrix}-[R+(n+r)\cos\theta]\sin\phi & [R+(n+r)\cos\theta]\cos\phi & 0\\ -(n+r)\cos\phi\sin\theta & -(n+r)\sin\theta\sin\phi & (n+r)\cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

The generic formula for the gradient operator gives (Mathematica is handy in this step)

$$\nabla = \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \left([H][F]^{\mathrm{T}} \right)^{-1} \begin{Bmatrix} \partial / \partial\phi \\ \partial / \partial\theta \\ \partial / \partial n \end{Bmatrix} = \frac{1}{R+(n+r)\cos\theta} \vec{e}_\phi \frac{\partial}{\partial\phi} + \vec{e}_\theta \frac{1}{n+r} \frac{\partial}{\partial\theta} + \vec{e}_n \frac{\partial}{\partial n} \; .$$

Finally, curvature of the torus geometry becomes (at the mid-surface $n=0$)

$$\nabla_0 \vec{e}_n = \frac{1}{R+r\cos\theta} \vec{e}_\phi \frac{\partial \vec{e}_n}{\partial\phi} + \vec{e}_\theta \frac{1}{r} \frac{\partial \vec{e}_n}{\partial\theta} + \vec{e}_n \frac{\partial \vec{e}_n}{\partial n} \quad \Rightarrow$$

$$\overleftrightarrow{\kappa} = (\nabla_0 \vec{e}_n)_{\mathrm{c}} = \frac{\cos\theta}{R+r\cos\theta} \vec{e}_\phi \vec{e}_\phi + \vec{e}_\theta \vec{e}_\theta \frac{1}{r} \; .$$