# Lecture 3

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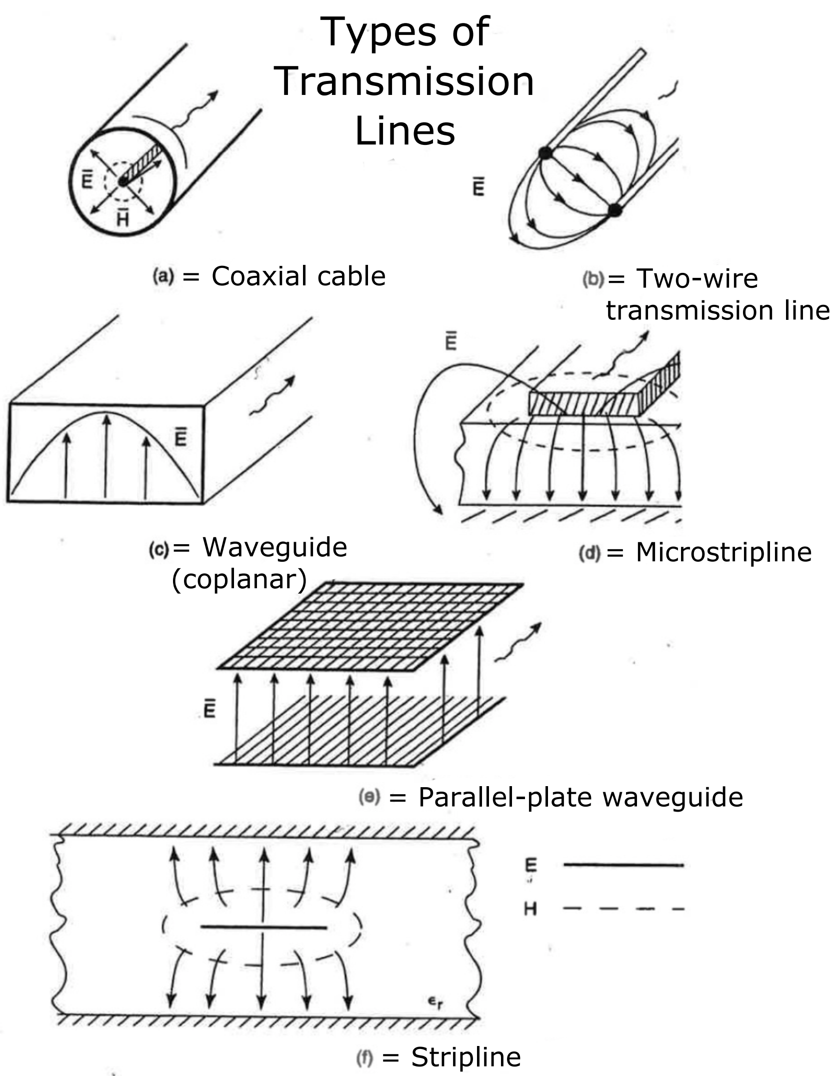
## I. TRANSMISSION LINES

– Electromagnetic waves can propagate in free space (Review this! Based on Maxwell's equations!). But also they can be guided by conducting or dielectric boundaries.

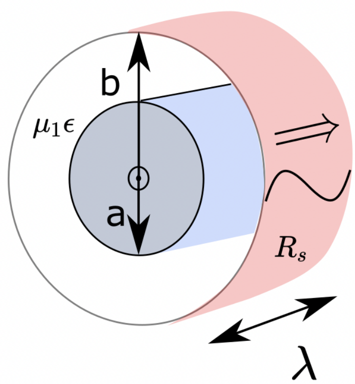
– Transmission line behavior: occurs when $\lambda \ll$ length of transmission line.

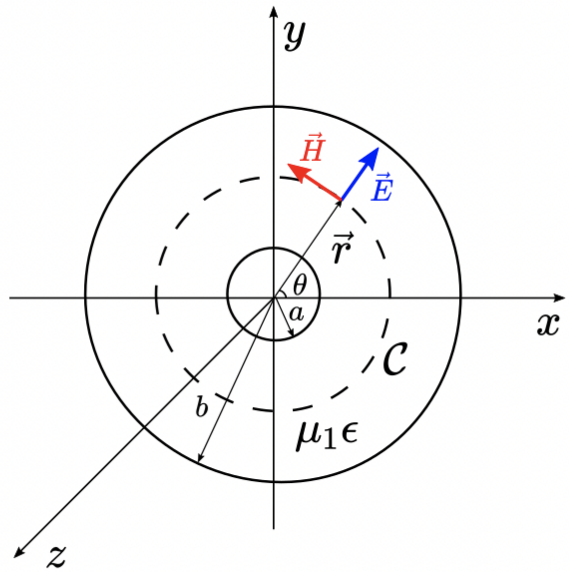
– Transmission lines = guiding devices for the electromagnetic field.

– The electromagnetic fields are TEM (transverse electromagnetic mode) if the conductors are ideal (zero-resistance); otherwise there will be a small axial component of the electromagnetic field.



EXAMPLE: The coaxial line



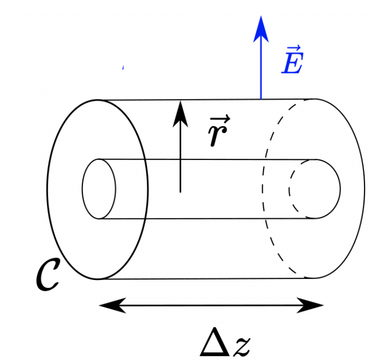

* How to calculate the $\vec{E}$, $\vec{H}$ fields inside?

$$\text{Electric Field:} \qquad \vec{E} = \frac{V_0}{\ln\frac{b}{a}}\frac{\hat{\vec{r}}}{r} \tag{1}$$

Proof:

$\vec{\nabla}\cdot\vec{D} = \rho \implies \int d\vec{S}\cdot\vec{E} = \int \frac{\rho}{2} dV \implies 2\pi r\cdot(\Delta z)\cdot E = \frac{1}{\epsilon}(\Delta z)\cdot\rho\cdot\pi a^2 \quad \therefore E = \frac{1}{r}\cdot\frac{\rho a^2}{2\epsilon}$

Also $V_0 = \int_a^b dr\cdot E = \int_a^b \frac{dr}{r}\cdot\rho\frac{a^2}{2\epsilon}\ln\frac{b}{a} \to \rho\frac{a^2}{2\epsilon} = \frac{V_0}{\ln\frac{b}{a}}$, so $\vec{E} = \frac{V_0}{\ln\frac{b}{a}}\frac{\hat{\vec{r}}}{r}$.



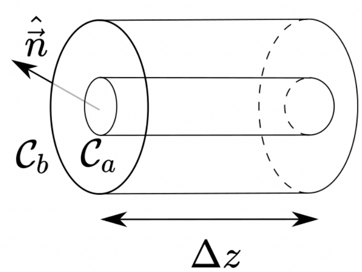

$$\text{Magnetic Field:} \qquad \vec{H} = \frac{I_0}{2\pi r}\cdot\hat{\vec{e_\theta}} \tag{2}$$

Proof:

$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \implies \int_{\mathcal{C}} \vec{H}\cdot d\vec{\ell} = \int \vec{J}\cdot d\vec{S} = I_0$, or $2\pi r\cdot H = I_0 \implies \vec{H} = \frac{I_0}{2\pi r}\cdot\hat{\vec{e_\theta}}$.

## II. TOWARDS A DISTRIBUTED MODEL OF INDUCTORS, CAPACITANCES, RESISTANCES, CONDUCTANCES

Problem: How to connect the electric and magnetic fields to circuit elements.




Answer: Via stored or dissipated energy.

1. Inductance per unit length

   Magnetic energy $= \frac{\mu}{4}\int ds \cdot (\Delta z)\vec{H}^2 = \frac{(L'\Delta t)I_0^2}{4} \implies L' = \frac{\mu}{I_0^2}\int ds\vec{H}^2$

   $L' = \frac{\mu}{I_0^2}\int dsH^2 = \frac{\mu}{I_0^2}\cdot I_0^2\int_0^{2\pi}d\theta\int_a^b dr\cdot r\cdot\frac{1}{(2\pi r)^2} = \frac{\mu}{2\pi}\ln\frac{b}{a}.$

   Therefore,
$$L' = \frac{\mu}{2\pi}\ln\frac{b}{a} \quad \text{(measured in units of H/m)}\ . \tag{3}$$

2. Capacitance per unit length

   Electrostatic energy$= \frac{\epsilon}{4}\int ds\cdot(\Delta z)\cdot E^2 = \frac{(C'\Delta z)V_0^2}{4} \implies C' = \frac{\epsilon}{V_0^2}\cdot\int ds\cdot E^2$

   $C' = \frac{\epsilon}{V_0^2}\int dsE^2 = \frac{\epsilon}{V_0^2}\cdot V_0^2\cdot\frac{1}{\ln^2\frac{b}{a}}\int_0^{2\pi}d\theta\int_0^b dr\cdot r\cdot\frac{1}{r^2}$
$$\implies \qquad C' = \frac{2\pi\epsilon}{\ln\frac{b}{a}} \quad \text{(measured in units of F/m)}\ . \tag{4}$$

3. Resistance per unit length

   Power dissipated in the lossy conductors $= \frac{R_s}{2}\int_{\mathcal{C}_a+\mathcal{C}_b}d\ell\cdot\Delta z\cdot\mathcal{J}_s^2 = \frac{R_s}{2}\Delta z\cdot\int_{\mathcal{C}_a+\mathcal{C}_b}d\ell\cdot H^2 = \frac{R'\Delta z}{2}I_0^2$. Here $R_s$ = surface resistance, $\vec{\mathcal{J}}_s = \hat{\vec{n}}\times\vec{H}$ = surface current, $\hat{\vec{n}}$ = vector unit pointing outwards (normal to the conducting surface), and $R' = \frac{R_s}{I_0^2}\int_{\mathcal{C}_a+\mathcal{C}_b} d\ell \cdot \vec{H}^2$.

$R' = \frac{R_s}{I_0^2}\int_{\mathcal{C}_a+\mathcal{C}_b} d\ell \cdot H^2 = \frac{R_s}{(2\pi)^2}\left[\int_0^{2\pi} d\theta \cdot a \cdot \frac{1}{a^2} + \int_0^{2\pi} d\theta \cdot b \cdot \frac{1}{b^2}\right] = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$

$$R^{'} = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) \quad \text{(measured in units of } \Omega/\text{m)} . \tag{5}$$

4. <u>Conductance (radial) per unit length</u>

$\epsilon = \epsilon^{'} - i\epsilon^{''} = \epsilon_0\epsilon_r(1 - i\tan\delta)$

$\epsilon' = \epsilon_0\epsilon_r$

$\underline{\epsilon^{''} = \epsilon\tan\delta} \rightarrow$ dissipation in the dielectric between the core metal and the outside shield.

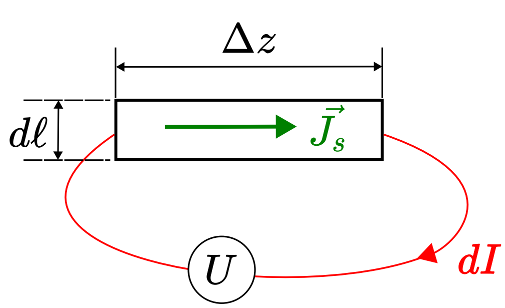
Power dissipated $= \frac{\omega\epsilon^{''}}{2}\int ds \cdot \Delta z \cdot E^2 = \frac{G' V_0^2}{2} \rightarrow G'' = \frac{\omega\epsilon^{''}}{V_0^2}\int ds \cdot E^2$

$\Longrightarrow \; G' = \frac{\omega\epsilon^{''}}{V_0^2}\int ds \cdot E^2 = \frac{\omega\epsilon^{''}}{V_0^2} \cdot \int_0^{2\pi} d\theta \int_a^b dr \cdot r \cdot \frac{V_0^2}{r^2 \ln\frac{b}{a}}$

$$\Longrightarrow \qquad G' = \frac{2\pi\omega\epsilon^{''}}{\ln\frac{b}{a}} \quad \text{(measured in units of S/m)} . \tag{6}$$

**NOTE:** In the calculation 3. resistance per unit length we used the surface resistance $R_s$ and the surface current $\vec{\mathcal{J}}_s$. Note that these concepts are different from "just" resistance and from the current density $\mathcal{J}$ which appears in Maxwell's equation (see Lecture 1) or when we write Ohm's law in the form $\vec{\mathcal{J}} = \sigma\vec{E}$ (see Lecture 2).

Consider a slab of metal, like in the next figure. The current density is defined as $\mathcal{J} = \frac{dI}{ds}$,




so it represents the current per unit area that flows through the wires. It has dimension of (charge/unit time)/(unit area), so it is measured in A/m$^2$. But for a thin sheet of metal a sensible quantity to define is

$$\mathcal{J}_s = \frac{dI}{dl} \tag{7}$$

which means (charge/unit time)/(unit length across the sheet). It is called surface current density and it is measured in A/m.

We can then consider the quantity $U/(\Delta z)$, which is the drop of voltage per unit lenght in the direction of the current, and define

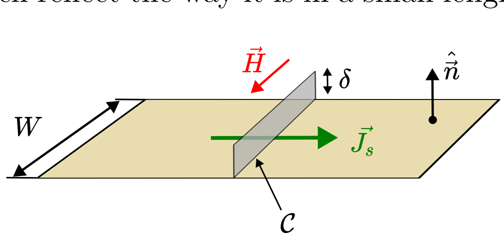
$$R_s = \frac{U/(\Delta z)}{\mathcal{J}_s} \tag{8}$$

This quantity has units of Ohms, so we can call it resistance. As a result, the power dissipated is

$$\frac{1}{2}\int dI \cdot U = \frac{1}{2}\Delta z \int_{\mathcal{C}} dl (R_s \mathcal{J}_s)\mathcal{J}_s = \frac{1}{2}\Delta z \int_{\mathcal{C}} dl R_s \mathcal{J}_s^2 \tag{9}$$

where $\mathcal{C}$ is a contour containing the element $dl$.

To obtain the connection between the surface current density and the magnetic field, consider the contour $\mathcal{C}$ in the figure below. We also take for simplicity the vector $\vec{H}$ as shown in the figure, which reflect the way it is in a small lenght $W$ around the coaxial cable.




So we have from the Maxwell-Ampére law $\nabla \times \vec{H} = \vec{\mathcal{J}} + \frac{\partial \vec{D}}{\partial t}$ and therefore applying Stokes' theorem we get

$$\int_{\mathcal{C}} d\vec{l}\vec{H} = HW \tag{10}$$

$$= \int_S d\vec{s}\left(\vec{\mathcal{J}} + \frac{\partial \vec{D}}{\partial t}\right) = \mathcal{J}\delta W = \mathcal{J}_s W \tag{11}$$

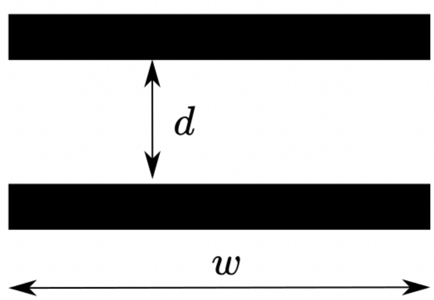
where you can see using the definitions above that $\mathcal{J}_s = \mathcal{J}\delta$. So in the end $\mathcal{J}_s = H$. It is easy to generalize this relation to $\vec{\mathcal{J}}_s = \hat{\vec{n}} \times \vec{H}$.

- Examples of materials used in coaxes:

| **Conductor** | Copper Cu | Aluminum Al | Silver Ag | Gold Au |
|---|---|---|---|---|
| **Resistivity** $\rho[\mathrm{n}\Omega \cdot \mathrm{m}]$ | 16.9 | 26.7 | 16.3 | 22.0 |

| **Dielectric** | Dry Air | Polyethylene | PTFE | PVC |
|---|---|---|---|---|
| $\epsilon_\mathbf{r}$ | 1.0006 | 2.2 | 2.1 | 3.2 |
| $\tan\delta$ | low | 0.0002 | 0.0002 | 0.001 |
| **Resistivity** $(\Omega \cdot \mathrm{m})$ | high | $10^{15}$ | $10^{15}$ | $10^{15}$ |
| **Breakdown voltage** $(\mathrm{mV/m})$ | 3 | 47 | 59 | 34 |

Other transmission lines:

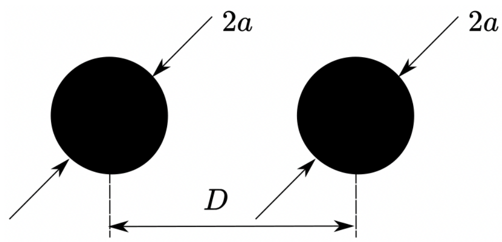



$$L' = \frac{\mu d}{w} \tag{12}$$

$$C' = \frac{\epsilon' w}{d} \tag{13}$$

$$R' = \frac{2R_s}{w} \tag{14}$$

$$G' = \frac{\omega \epsilon'' w}{d} \tag{15}$$




$$L' = \frac{\mu}{\pi} \cosh^{-1} \frac{D}{2a} \tag{16}$$

$$C' = \frac{\pi \epsilon'}{\cosh^{-1} \frac{D}{2a}} \tag{17}$$

$$R' = \frac{R_s}{\pi a} \tag{18}$$

$$G' = \frac{\pi \omega \epsilon''}{\cosh^{-1} \frac{D}{2a}} \tag{19}$$

## III. TRANSMISSION LINES: GENERAL MODELS

– If the length of a circuit is $\gtrsim \lambda$ we have to use either a simulator of Maxwell's equations or a distributed model of lumped elements.

– Transmission lines: Two parallel conductors that guide the electromagnetic field. Examples: two-wire lines, striplines, microstrip lines.

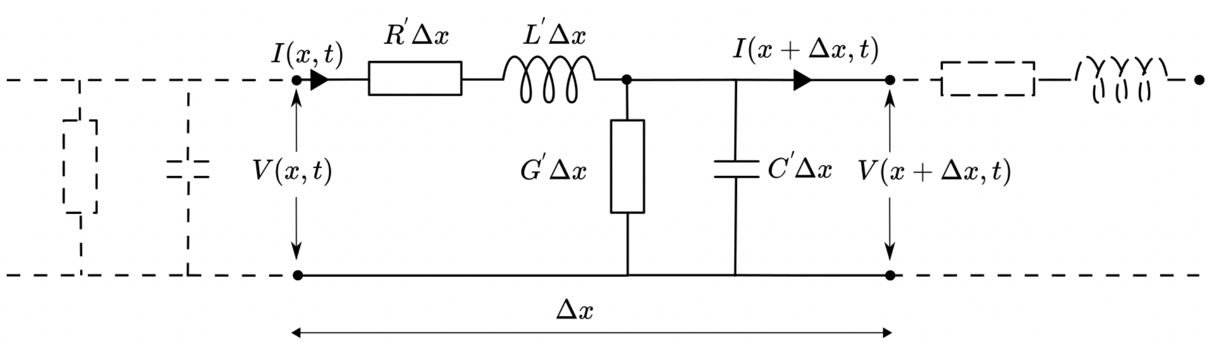



FIG. 1. Schematic of the distributed model of a transmission line, with lumped circuit elements defined per unit lenght $L', C', R', G'$. Here we denote the coordinate along the line by $x$ (instead of $z$ as in the previous section), in order to avoid any notational confusion with the $Z$ of impedances.

$R', L', G', C'$ = resistance, inductance, conductance, capacitance per unit length.

Kirchoff says:

$$\begin{cases} V(x,t) = I(x,t)R'\Delta x + L'\Delta x \cdot \frac{\partial I(x,t)}{\partial t} + V(x+\Delta x, t) \\ I(x,t) = V(x+\Delta x, t)G'\Delta x + C'\Delta x \frac{\partial V(x+\Delta x,t)}{\partial t} + I(x+\Delta x, t) \end{cases} \tag{20}$$

$$\Delta x \longrightarrow 0 \begin{cases} -\frac{\partial V(x,t)}{\partial x} = R'I(x,t) + L'\frac{dI(x,t)}{dt} \\ -\frac{\partial I(x,t)}{\partial x} = G'V(x,t) + C'\frac{\partial V(x,t)}{\partial t} \end{cases} \tag{21}$$

Therefore,

$$\begin{cases} -\frac{\partial^2 V(x,t)}{\partial x^2} = -R'(G'V(x,t) + C'\frac{\partial V(x,t)}{\partial t}) - L'(G'\frac{\partial V(x,t)}{\partial t} + C'\frac{\partial^2 V(x,t)}{\partial t^2}) \\ -\frac{\partial^2 I(x,t)}{\partial x^2} = -G'(R'I(x,t) + L'\frac{\partial I(x,t)}{\partial t}) - C'(R'\frac{\partial I(x,t)}{\partial t} + L'\frac{\partial^2 I(x,t)}{\partial t^2}) \end{cases} \tag{22}$$

or

$$\begin{cases} \frac{\partial^2 V(x,t)}{\partial x^2} = L'C'\frac{\partial^2 V(x,t)}{\partial t^2} + (R'C' + L'G')\frac{\partial V(x,t)}{\partial t} + R'G'V(x,t) \\ \frac{\partial^2 I(x,t)}{\partial x^2} = L'C'\frac{\partial^2 I(x,t)}{\partial t^2} + (R'C' + L'G')\frac{\partial I(x,t)}{\partial t} + R'G'I(x,t)\ . \end{cases} \tag{23}$$

<u>Harmonic signals:</u>

$V(x,t) = V(x)e^{i\omega t}$ , $\qquad V(x), I(x)$ = phasors, $\qquad I(x,t) = I(x)e^{i\omega t}$

$$\Longrightarrow \begin{cases} \frac{d^2V(x)}{dx^2} - \gamma^2 V(x) = 0, \quad \text{where} \quad \gamma = \alpha + i\beta = \sqrt{(R' + i\omega L')(G' + i\omega C')} \\ \frac{d^2 I(x)}{dx^2} - \gamma^2 I(x) = 0, \quad \gamma = \text{propagation constant},\ \alpha = \text{attenuation constant},\ \beta = \text{phase constant} \end{cases}$$

$$\text{General Solution:} \qquad V(x) = V^{\dagger}e^{-\gamma x} + V^{-}e^{\gamma x}\ . \tag{24}$$

From $-\frac{\partial V(x,t)}{\partial x} = R'I(x,t) + L'\frac{\partial I(x,t)}{\partial t}$, we get $I(x) = -\frac{1}{R'+i\omega L'}\frac{dV(x)}{dx}$ or

$I(x) = \frac{1}{Z_0}V^{\dagger}e^{\gamma x} - \frac{1}{Z_0}V^{-}e^{\gamma x} = I^{+}e^{-\gamma x} + I^{-}e^{\gamma x}$, where

$Z_0 = \sqrt{\frac{R'+i\omega L'}{G'+i\omega C'}}$ = characteristic impedance of the transmission line,

and where $I^{\pm} = \pm\frac{V^{\pm}}{Z_0}$.

Lossless transmission case: $R' = G' = 0$

$\gamma = i\beta = i\omega\sqrt{L'C'}$

$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L'}{C'}} \longrightarrow$ now independent of frequency!
Note: Free-space impedance = 377 Ω
$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$ = phase velocity.

Exercise: Show that for the loss-less case $R' \ll \omega L'$, $G' \ll \omega C'$, we have $\beta \simeq \omega\sqrt{L'C'}$ and $\alpha \simeq \frac{1}{2}\sqrt{L'C'}(\frac{R'}{L'} + \frac{G'}{C'})$.

TABLE I. Some standardized values of $Z_0$.

| $Z_0$ | **Application** |
|---|---|
| 50 Ω | Instrumentation, communication |
| 75 Ω | TV, VHF radio |
| 300 Ω | RF |
| 600 Ω | Audio |

**NOTE:**

Let's contemplate a bit what is the meaning of the quantities $V^+e^{-\gamma x}$ and $V^-e^{\gamma x}$. These are the phasors of the waves propagating in the positive and negative direction of the $x$-axis, respectively.

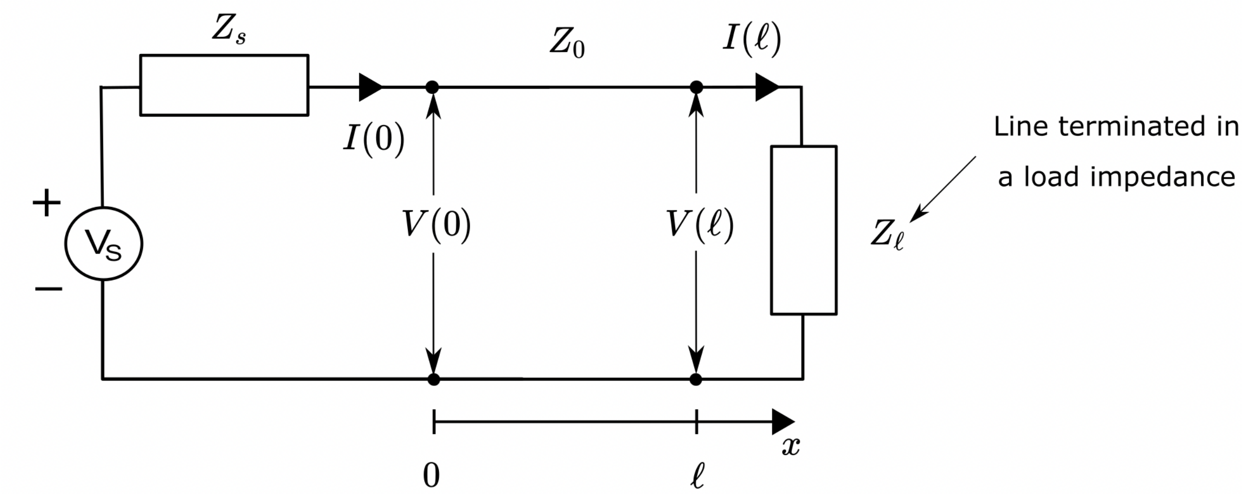
How do you see this?

Let's take first the "+" wave. We have

$V^+e^{-\gamma x}e^{i\omega t} = V^+e^{-\alpha x}e^{i\omega\left(t-\frac{x}{v_p}\right)}$, where $\gamma = \alpha + i\beta$ and the phase velocity $v_p = \frac{\omega}{\beta}$. Now let's look at what happens of we follow a constant phase $t - \frac{x}{v_p} = \varphi$. We get $x = v_p t - \varphi$. So this means propagation in the positive $x$ -direction.

In contrast, the "-" wave

$V^-e^{\gamma x}e^{i\omega t} = V^-e^{-\alpha x}e^{i\omega\left(t+\frac{x}{v_p}\right)}$, where also here $\gamma = \alpha + i\beta$ and the phase velocity $v_p = \frac{\omega}{\beta}$. Again we follow a constant phase but now that phase is $t + \frac{x}{v_p} = \varphi$. We get $x = -v_p t - \varphi$. So this means propagation in the negative $x$ -direction.




## IV. INCIDENT AND REFLECTED WAVES ALONG A LOADED TRANSMISSION LINE

$V(x) = V^+e^{-\gamma x} + V^-e^{\gamma x}$

$I(x) = I^+e^{-\gamma x} + I^-e^{\gamma x}, \quad I^\pm = \pm\frac{V^\pm}{Z_0}$

$$\begin{cases} V(0) = V_s - Z_s I_0 \text{ — Kirchoff's law} \\ V(\ell) = Z_\ell I(\ell) \ , \end{cases} \tag{25}$$

or

$$\begin{cases} V^+ + V^- = V_s - \frac{Z_s}{Z_0}(V^+ + V^-) \\ V^+e^{-\gamma\ell} + V^-e^{\gamma\ell} = \frac{Z_\ell}{Z_0}(V^+e^{-\gamma\ell} - V^-e^{\gamma\ell}) \end{cases} \tag{26}$$

- Reflection and transmission coefficients

  Define a reflection coefficient of the load at $x = \ell$: $\Gamma_V = \frac{V^-e^{\gamma\ell}}{V^+e^{-\gamma\ell}}$.

  $\to 1 + \Gamma_V = \frac{Z_\ell}{Z_0}(1 - \Gamma_V)$.

$$\Longrightarrow \qquad \Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} \tag{27}$$

  We can also define a current reflection coefficient at the load

$$\Gamma_I = \frac{I^-e^{\gamma\ell}}{I^+e^{-\gamma\ell}} = -\Gamma_V \tag{28}$$

Define a transmission coefficient at the load $x = \ell$: $T_V = \frac{V^+ e^{+\ell} + V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$.

$$\therefore \qquad T_V = 1 + \Gamma_V \ , \tag{29}$$

and for the current

$$T_I = \frac{I^+ e^{-\gamma\ell} + I^- e^{\gamma\ell}}{I^+ e^{-\gamma\ell}} = 1 + \Gamma_I \ . \tag{30}$$

- Average power delivered to the load

$\overline{P_\ell} = \frac{1}{2}\mathrm{Re}[V(\ell)I^*(\ell)]$, where the 1/2 comes from the fact that the field is harmonic.

Now,

$$\begin{cases} 1 - \Gamma_V = \frac{I^- e^{\gamma\ell} + I^+ e^{-\gamma\ell}}{I^+ e^{-\gamma\ell}} = \frac{I(\ell)}{I^+ e^{-\gamma\ell}} \\ 1 + \Gamma_V = \frac{V^+ e^{-\gamma\ell} + V^- e^{-\gamma\ell}}{V^+ e^{\gamma\ell}} = \frac{V(\ell)}{V^+ e^{-\gamma\ell}} \ . \end{cases} \tag{31}$$

$\therefore \qquad (1 + \Gamma_V^*)(1 + \Gamma_V) = \frac{V(\ell)I^*(\ell)}{I^{+*}V^+ e^{-\gamma\ell}(e^{-\gamma\ell})^*}$, but $I^+ = \frac{V^+}{Z_0}$

$V(\ell)I^*(\ell) = \frac{1}{Z_0}|V^+ e^{-\gamma\ell}|^2 \cdot (1 - \Gamma_V^*)(1 + \Gamma_V) \equiv 1 - \Gamma_V^* + \Gamma_V - |\Gamma_V|^2$ , where $\Gamma_V^* + \Gamma_V =$ Imaginary!.

$$\overline{P_\ell} = \frac{1}{2Z_0} \cdot |V^+ e^{-\gamma\ell}|^2 (1 - |\Gamma_V|^2) \ . \tag{32}$$

- VSWR (Voltage standing-wave ratio)

$V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x} = V^+ e^{-\gamma x}[1 + \Gamma_V e^{-2\gamma(\ell - x)}]$ (Remember that $\Gamma_V \equiv \frac{V^- e^{\gamma\ell}}{V^+ e^{-\gamma\ell}}$.)

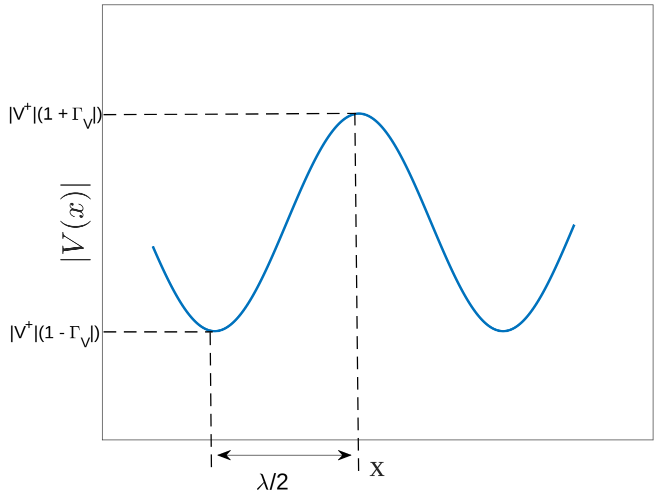
Let's consider a lossless line $\alpha = 0, \quad \gamma = i\beta = \frac{2\pi i}{\lambda}$

$|V(x)| = |V^+| \cdot |1 + \Gamma_V e^{-2i\beta(\ell - x)}|$ — oscillates, min. and max. separated by $\frac{\pi}{\beta} = \frac{\lambda}{2}$.

$VSWR = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|}$ = ratio between the max. line voltage and min. line voltage.

- Impedance along the line

$Z(x) = \frac{V(x)}{I(x)} = Z_0 \frac{V^+ e^{-\gamma x} + V^- e^{\gamma x}}{V^+ e^{-\gamma x} - V^- e^{\gamma x}} = \frac{1 + \Gamma_V e^{-2\gamma(\ell - x)}}{1 - \Gamma_V e^{-2\gamma(\ell - x)}}$.




Take $x = 0 \rightarrow$ we get $Z(0) \equiv Z_{in}$ = input impedance of the line, i.e., the impedance seen when looking toward the load.

$$Z_{in} = Z_0 \cdot \frac{Z_\ell + Z_0 \tanh \gamma\ell}{Z_0 + Z_\ell \tanh \gamma\ell} \tag{33}$$

Note that this can be verified immediately by recalling that $\Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0}$, and that in general, $Z_{in} \neq Z_0$, so the termination matters! Also, $Z_{in}$ is frequency-dependent.

## V. FURTHER READING


- David M. Pozar — Microwave Engineering.
- R.E. Collin — Foundations for Microwave Engineering.